Quantum dynamics of a two-level atom. Rabi oscillations

The dynamics of a two-level systems is a widely encountered problem in many applications. Some systems are inherently two-level ones (e.g. the spin of a spin-1/2 particle). For some others a two-level approximation can serve as a good model that in certain cases can capture qualitative behavior very well.

Let us consider a (quantum) atom whose energy levels are $E_1 = \hbar\omega_1$ and $E_2 = \hbar\omega_2$ $(E_2 > E_1)$, so that the angular transition frequency is $\omega_{21} = \omega_2 - \omega_1$. As we saw it in one of the previous lectures, the dynamics of such system is described by two coupled differential equations

$$\dot{c}_1 = -\frac{i}{\hbar}\left[c_1 H'_{11} + c_2 H'_{12} e^{-i\omega_{21}t}\right] \qquad (*)$$

$$\dot{c}_2 = -\frac{i}{\hbar}\left[c_1 H'_{21} e^{i\omega_{21}t} + c_2 H'_{22}\right]$$

where $H'_{ij} = \langle \psi_i | H'(t) | \psi_j \rangle$, etc.

(Recall that in our previous notations $H = H_0 + H'(t)$

$H_0\psi_1 = E_1\psi_1 \qquad H_0\psi_2 = E_2\psi_2$)

Equation $(*)$ makes no assumption regarding the strength of $H'(t)$. Matrix elements H'_{12} are functions of time, but we assume we can compute them for any t.

Now let us consider the dynamics of our two-level atom under the specific form of H':

$$H'(t) = ex\mathcal{E}\cos\omega t$$

Unlike in the case of perturbative treatment, we do not require that $\mathcal{E}$ is weak.

As we found out in the previous lecture on Selection rules, the diagonal elements of H', namely

$$H'_{11} = \langle\psi_1|x|\psi_1\rangle e\mathcal{E}\cos\omega t$$

$$H'_{22} = \langle\psi_2|x|\psi_2\rangle e\mathcal{E}\cos\omega t,$$

vanish because atomic eigenstates have definite parity (even or odd) and the dipole operator ex is of odd parity.

The off-diagonal matrix elements for bound atomic states are real quantities (or can be made real) and are

$$H'_{21} = H'_{12} = \langle\psi_2|x|\psi_1\rangle e\mathcal{E}\cos\omega t$$

We define the Rabi frequency Ω as

$$\Omega = \frac{e\mathcal{E}}{\hbar}\langle\psi_1|x|\psi_2\rangle$$

With that our system of two coupled equations for c_a and c_b reduces to

$$\dot{c}_1 = -i\Omega\cos\omega t\, e^{-i\omega_{21}t}\, c_2$$

$$\dot{c}_2 = -i\Omega\cos\omega t\, e^{i\omega_{21}t}\, c_1$$

We can make use of the relation $\cos\omega t = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})$

and rewrite the equations as

$$\dot{c}_1 = -\frac{1}{2} i \Omega \left[e^{i(\omega-\omega_{21})t} + e^{-i(\omega+\omega_{21})t} \right] c_2$$

$$\dot{c}_2 = -\frac{1}{2} i \Omega \left[e^{i(\omega+\omega_{21})t} + e^{-i(\omega-\omega_{21})t} \right] c_1$$

Now, this system of two equations cannot be solved easily (analytically). However, if we neglect the rapidly oscillating term $e^{-i(\omega+\omega_{6a})t}$, i.e. if we assume that $|\omega-\omega_{21}| << \omega+\omega_{21}$ (this is often called the rotating wave approximation. - RWA) then we can solve the resulting equations:

$$\dot{c}_1 = -\frac{i\Omega}{2} e^{i(\omega-\omega_{21})t} c_2$$

$$\dot{c}_2 = -\frac{i\Omega}{2} e^{-i(\omega-\omega_{21})t} c_1$$

To simplify notations we will the so called detuning: $\delta \equiv \omega - \omega_{21}$

With that we get

$$\dot{c}_1 = -\frac{i\Omega}{2} e^{i\delta t} c_2$$

$$\dot{c}_2 = -\frac{i\Omega}{2} e^{-i\delta t} c_1$$

From the last equation we have $c_1 = \frac{2i}{\Omega} e^{i\delta t} \dot{c}_2$, or if we differentiate it with respect to t,

$$\dot{c}_1 = -\frac{2\delta}{\Omega} e^{i\delta t} \dot{c}_2 + \frac{2i}{\Omega} e^{i\delta t} \ddot{c}_2$$

When we substitute it in the equation for $\dot{c}_1$ we obtain

$$\ddot{c}_2 + i\delta \dot{c}_2 + \frac{\Omega^2}{4} c_2 = 0$$

This is an ordinary differential equation of second order that we know how to solve:

$$\lambda^2 + i\delta\lambda + \frac{\Omega^2}{4} = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left[-i\delta \pm \sqrt{-\delta^2 - \Omega^2}\right] = i\left[-\frac{\delta}{2} \pm \omega_R\right]$$

where $\omega_R \equiv \frac{1}{2}\sqrt{\delta^2 + \Omega^2} = \frac{1}{2}\sqrt{(\omega - \omega_{21})^2 + \Omega^2}$ — Rabi flopping frequency

$$c_2(t) = A e^{i(-\frac{\delta}{2} + \omega_R)t} + B e^{i(-\frac{\delta}{2} - \omega_R)t} = e^{-i\frac{\delta t}{2}}\left[A e^{i\omega_R t} + B e^{-i\omega_R t}\right]$$

Alternatively, we can write it as

$$c_2(t) = e^{-\frac{i\delta t}{2}}\left[F \cos\omega_R t + G \sin\omega_R t\right]$$

Now let us assume that at $t=0$ the atom was in state 1, i.e. $c_1(0) = 1$ $c_2(0) = 0$. Because of the latter we can conclude that $F=0$ and

$$c_2(t) = G e^{-\frac{i\delta t}{2}} \sin\omega_R t$$

To determine coefficient G we differentiate this expression

$$\dot{c}_2 = G\left[-i\frac{\delta}{2} e^{-i\frac{\delta t}{2}} \sin\omega_R t + \omega_R e^{-\frac{i\delta t}{2}} \cos\omega_R t\right]$$

then

$$c_1(t) = \frac{2i}{\Omega} e^{i\delta t} \dot{c}_2 = \frac{2i}{\Omega} G e^{\frac{i\delta t}{2}}\left[-\frac{i\delta}{2} \sin\omega_R t + \omega_R \cos\omega_R t\right]$$

Since $c_1(0) = 1$ we get

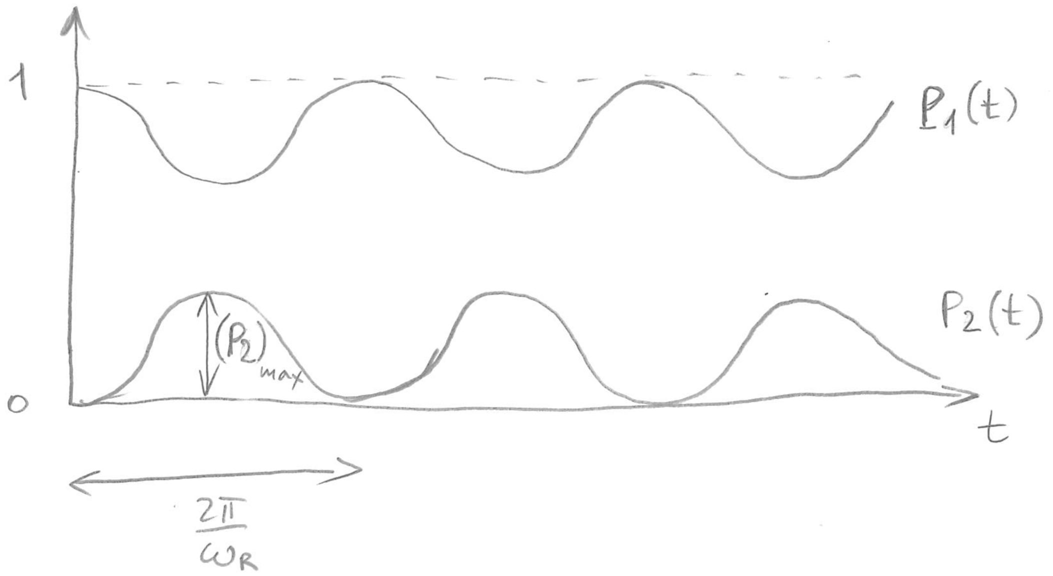
$$1 = \frac{2i}{\Omega} G \omega_R \quad \text{or} \quad G = -\frac{i\Omega}{2\omega_R}$$

So

$$C_2(t) = -\frac{i\Omega}{2\omega_R} e^{-\frac{i\delta t}{2}} \sin \omega_R t$$

and the corresponding probability of $1 \to 2$ transition is:

$$P_2(t) = |C_2(t)|^2 = \left(\frac{\Omega}{2\omega_R}\right)^2 \sin^2 \omega_R t = \underbrace{\frac{\Omega^2}{\delta^2 + \Omega^2}}_{(P_2)_{max}} \sin^2 \omega_R t$$



Rabi oscillations