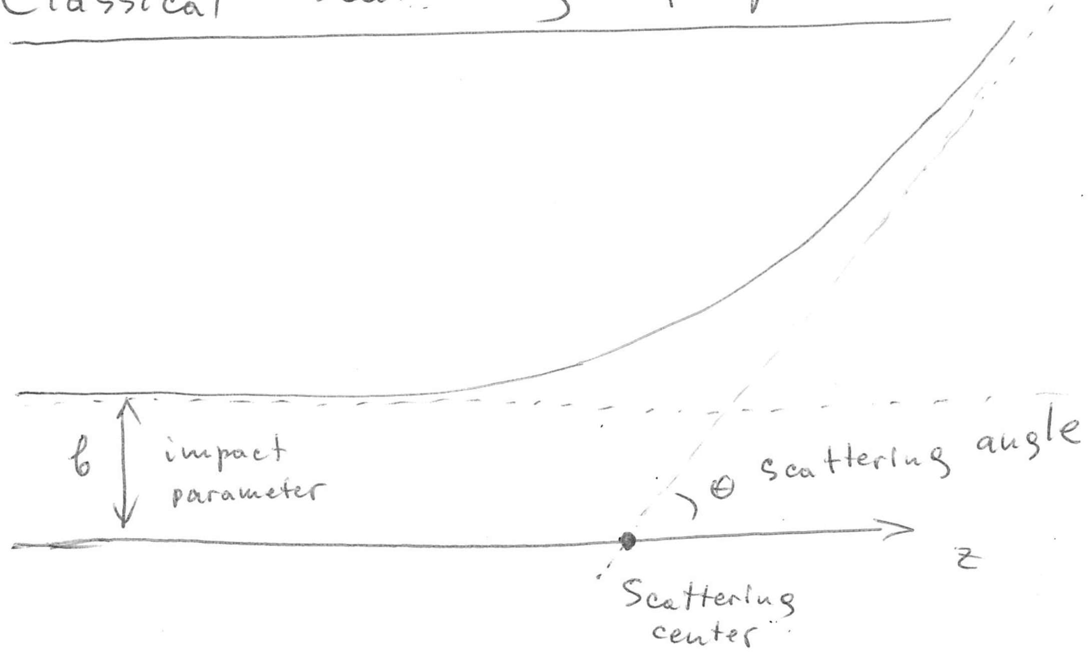
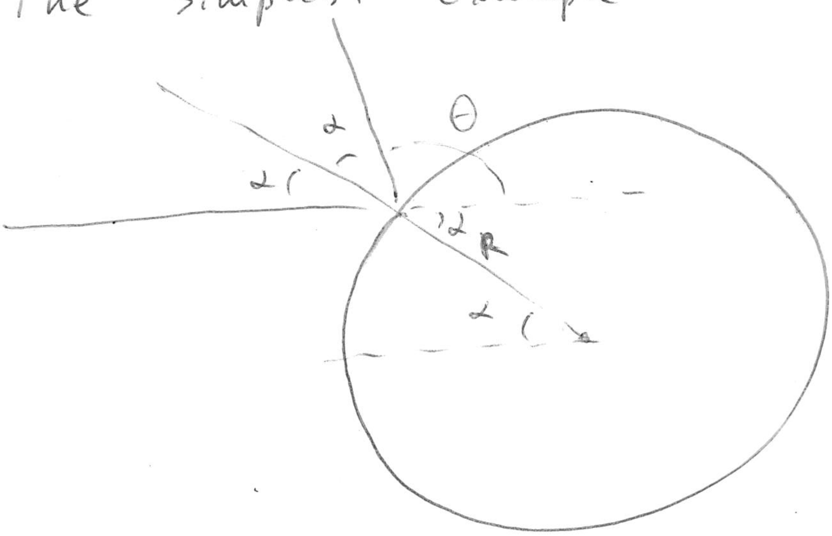
Classical scattering of particles



Let us consider a particle incident on some scattering center. The particle has some initial energy, E, and impact parameter, b. Eventually it scatters at some scattering angle θ. For simplicity we will assume that the target's potential is spherically symmetric. Then the trajectory lies in a single plane. Our task is to compute θ given b.

The simplest example is a hard-sphere potential



$$b = R\sin\alpha$$

$$\theta = \pi - 2\alpha$$

$$b = R\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = R\cos\frac{\theta}{2}$$

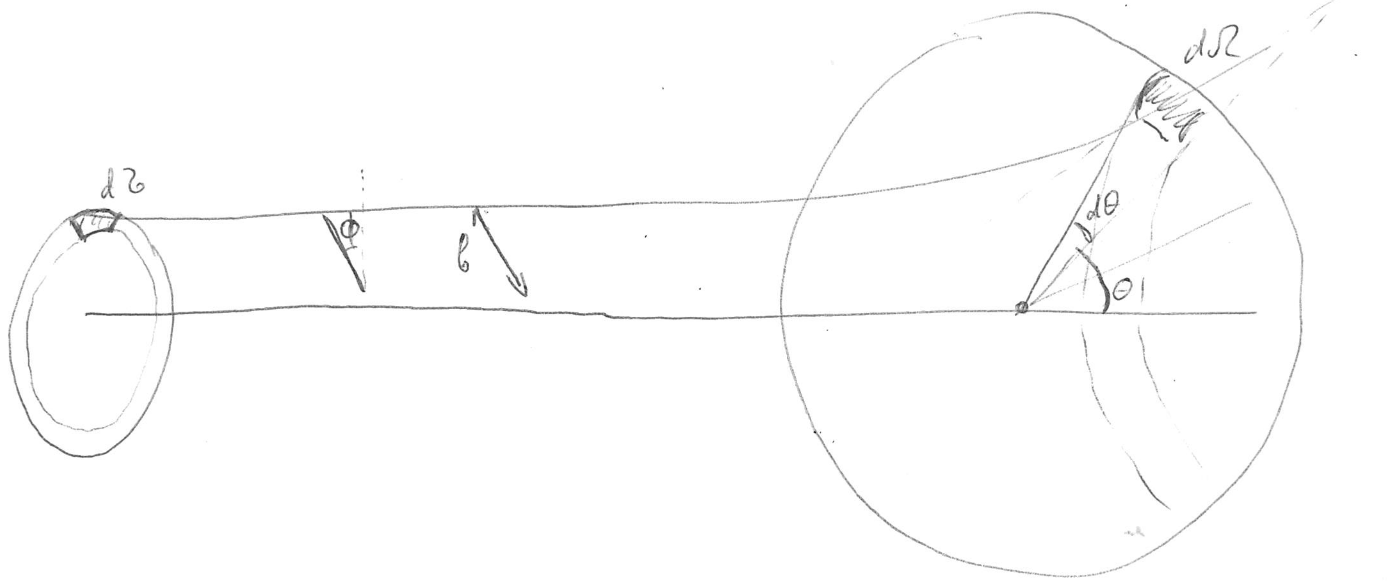
$$\theta = \begin{cases} 2\arccos\frac{b}{R}, & b < R \\ 0, & b \geq R \end{cases}$$

Differential cross section

$$D(\theta) = \frac{d\sigma}{d\Omega}$$

$d\sigma$ – crossectional area

$d\Omega$ – solid angle



$$d\sigma = D(\theta)\, d\Omega$$

$$d\sigma = b\, db\, d\phi \qquad d\Omega = \sin\theta\, d\theta\, d\phi$$

$$D(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

For a hard sphere

$$\frac{db}{d\theta} = -\frac{1}{2} R \sin\frac{\theta}{2}$$

and $$D(\theta) = \frac{R\cos\frac{\theta}{2}}{\sin\theta}\left(\frac{R\sin\frac{\theta}{2}}{2}\right) = \frac{R^2}{4}$$

The total cross-section is the integral of $D(\theta)$ over all solid angles:

$$\sigma \equiv \int D(\theta)\, d\Omega$$

For a hard sphere $$\sigma = \frac{R^2}{4}\int d\Omega = \pi R^2$$

If we have a beam of incident particles with uniform intensity (luminocity)

$\mathcal{L}$ - # particles per unit area per unit time

then the number of particles going through area $d\sigma$ (and scattered into solid angle $d\Omega$) per unit time is

$$dN = \mathcal{L}\, d\sigma = \mathcal{L}\, D(\theta)\, d\Omega$$

So

$$D(\theta) = \frac{1}{\mathcal{L}} \frac{dN}{d\Omega}$$

This is taken as the definition of the differential cross section because it makes reference only to quantities easily measured in the experiment.

As a model to practice let us consider a Coulomb system of charges q_1 and q_2 and masses $m_1 = m$ and $M_2 =$ infinity.

Conservation of energy gives $E = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\varphi}^2) + V(r)$

where $V(r) = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{1}{r}$

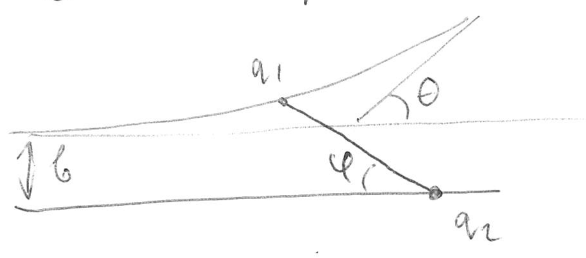
Conservation of angular momentum : $L = mr^2\dot{\varphi}$ So $\dot{\varphi} = \frac{L}{mr^2}$

$$\dot{r}^2 + \frac{L^2}{mr^2} = \frac{2}{m}(E - V)$$

let $u \equiv \frac{1}{r}$ then

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\varphi}\frac{d\varphi}{dt} =$$

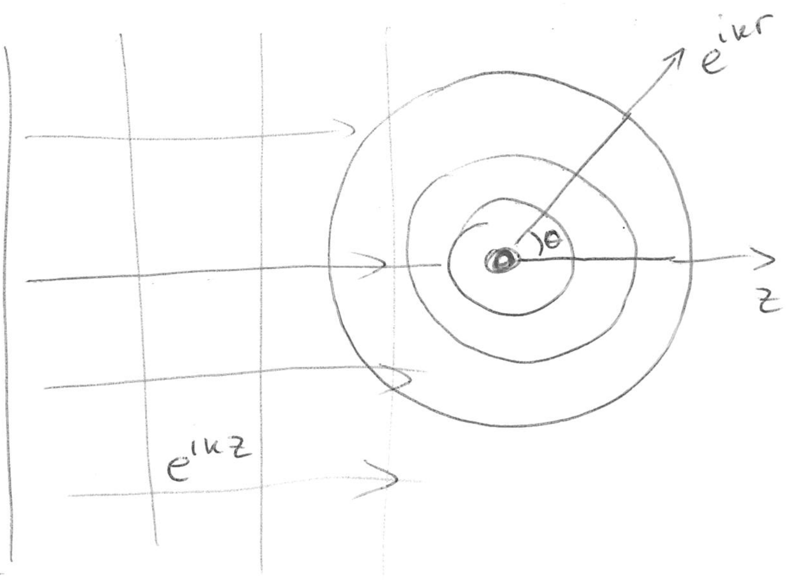
$$= -\frac{1}{u^2}\frac{du}{d\varphi}\frac{L}{m}u^2 = -\frac{L}{m}\frac{du}{d\varphi}$$



Scattering of waves (quantum scattering)

The general scattering problem in quantum mechanics is the calculation of $\frac{d\sigma}{d\Omega}$ for a particle incident on an arbitrary potential V. In principle, the Schrödinger (or equivalent) equation should be solved and the resulting wave function used to find the cross section. In practice, the SE for scattering problems is usually difficult to solve, so approximation methods must be used. There are two cases for which the problem becomes simpler. In the limit when $E \ll |V|$ the method of partial waves can be utilized. In the opposite limit, when $E \gg |V|$ it is possible to use a variant of the perturbation theory called the Born approximation. We will first consider the $E \ll |V|$ case. But before that let us consider some general properties of the scattering wave function.

Suppose we have an incident wave of particles traveling in the z-direction, Ae^{ikz}. When it encounters a scattering potential it produces an outgoing spherical wave. Hence we will look for solutions of the SE in the form



$$\psi(r,\theta,\phi) \underset{r\to\infty}{=} A\Big\{\underbrace{e^{ikz}}_{\text{incident}} + \underbrace{f(\theta,\phi)\frac{e^{ikr}}{r}}_{\text{scattered}}\Big\}$$

In the case of spherically symmetric potentials $V = V(|\vec{r}|)$ there will be no dependence on ϕ angle. Thus $f = f(\theta)$. Here $k = \frac{\sqrt{2mE}}{\hbar}$

The spherical wave contains $1/r$ factor because the total probability (which goes as $\underbrace{1/r^2}_{|\psi_{scattered}|^2} \cdot \underbrace{r^2}_{\text{area of sphere}}$) must be conserved.

The goal of the scattering theory is to determine $f(\theta)$ — the scattering amplitude as its square gives us the probability of scattering in a given direction θ.

Indeed the number of particles scattered into angle $d\Omega$ (which is in the direction of $\vec{e}_r$) is

$$\vec{J}_{incident} \cdot \vec{d\sigma} = r^2 J_{scattered,r}\, d\Omega$$

$$J_{scattered,r} = \frac{\hbar}{2mi}\left(\psi_{sc}^* \frac{\partial \psi_{sc}}{\partial r} - \psi_{sc}\frac{\partial \psi_{sc}^*}{\partial r}\right)$$

[recall $\vec{J} = \frac{\hbar}{2mi}(\psi^* \nabla\psi - \psi\nabla\psi^*)$ for Cartesian coordinates]

in spherical coordinates

$$\nabla\psi = \frac{\partial\psi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\vec{e}_\phi$$

$$\psi_{incident} = Ae^{ikz} \Rightarrow J_{incident} = \frac{\hbar k}{m}$$

$$J_{scattered,r} = \frac{\hbar k}{mr^2}|f(\theta)|^2$$

Therefore

$$r^2 \frac{\hbar k}{mr^2}|f(\theta)|^2 d\Omega = \frac{\hbar k}{m} d\sigma$$

and

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

The problem of determining $\frac{d\sigma}{d\Omega}$ (differential cross section) is equivalent to finding the scattering amplitude $f(\theta)$.