1 a) In our case $l=0$ $s=1/2$. The rules of addition of two angular momenta give us only one possibility: $j = 1/2$

b) Here $s = 1/2 \; (j = 1/2)$ and $i = 3/2$. Adding two angular momenta gives two possibilities:

$$f = 1 \qquad f = 2$$

c) The perturbing Hamiltonian can be written as

$$H_{hf} = A\vec{S}\cdot\vec{I} = \frac{A}{2}\left(F^2 - S^2 - I^2\right)$$

One could come up with these basis sets in this problem: $|l m_l s m_s i m_i\rangle$, $|lsjm_j i m_i\rangle$, $|lsjifm_f\rangle$

Clearly the latter one is most convenient as it is composed of simultaneous eigenfunctions of $F^2$, $S^2$, and $I^2$. An operator is diagonal in the basis of its eigenfunctions

d) $$\langle lsjifm_f | H_{hf} | lsjifm_f\rangle = \frac{A}{2}\langle lsjifm_f | F^2 - S^2 - I^2 | lsjifm_f\rangle =$$

$$= \frac{A\hbar^2}{2}\Big[f(f+1) - \underbrace{s(s+1)}_{3/4} - \underbrace{i(i+1)}_{15/4}\Big] = \frac{A\hbar^2}{2}\left[f(f+1) - 9/2\right]$$

Given that $f$ can take two values $f = 1, 2$ we have the following shifts:

$$E^{(1)}_{f=1} = -\frac{5}{4}A\hbar^2 \quad \leftarrow \text{three-fold degenerate}: m_f = -1, 0, +1$$

$$E^{(1)}_{f=2} = \frac{3}{4}A\hbar^2 \quad \leftarrow \text{five-fold degenerate}: m_f = -2, -1, 0, +1, +2$$

(2) A particle of mass $m$ has the following energy levels and wavefunctions in the box $0 \leq x \leq a$ $0 \leq y \leq a$ $0 \leq z \leq 2a$

$$E_{n_x n_y n_z} = \frac{\hbar^2 \pi^2}{2ma^2}\left(n_x^2 + n_y^2 + \frac{n_z^2}{4}\right) \qquad n_x = 1,2,\dots \quad n_y = 1,2\dots \quad n_z = 1,2\dots$$

$$\psi_{n_x n_y n_z} = \frac{2}{a^{3/2}} \sin\frac{\pi n_x x}{a} \sin\frac{\pi n_y y}{a} \sin\frac{\pi n_z z}{2a} = \phi_{n_x}(x)\, \phi_{n_y}(y)\, \varphi_{n_z}(z)$$

The lowest energy levels are $\left( A \equiv \frac{\hbar^2\pi^2}{2ma^2} \right)$

| $n_x$ | $n_y$ | $n_z$ | |
|---|---|---|---|
| 1 | 1 | 1 | $E = 2.25A$ |
| 1 | 1 | 2 | $E = 3A$ |
| 1 | 1 | 3 | $E = 4.25A$ |
| 2 | 1 | 1 | $E = 5.25A$, two-fold degenerate |
| 1 | 2 | 1 | |

So in the case of the fourth excited state we deal with two degenerate states:

$$|1\rangle \equiv |211\rangle$$
$$|2\rangle \equiv |121\rangle$$

The matrix elements of $V = \beta xyz$ in this basis are:

$$\langle 1|V|1\rangle = \beta \underbrace{\langle \phi_2(x)|x|\phi_2(x)\rangle}_{a/2} \underbrace{\langle \phi_1(y)|y|\phi_1(y)\rangle}_{a/2} \underbrace{\langle \varphi_1(z)|z|\varphi_1(z)\rangle}_{a} = \frac{a^3}{4}$$

$$\langle 2|V|1\rangle = \beta \underbrace{\langle \phi_2(x)|x|\phi_1(x)\rangle}_{-\frac{16a}{9\pi^2}} \underbrace{\langle \phi_1(y)|y|\phi_2(y)\rangle}_{-\frac{16a}{9\pi^2}} \underbrace{\langle \varphi_1(z)|z|\varphi_1(z)\rangle}_{a} = \frac{256}{81\pi^4}a^3$$

$$\langle 1|V|2\rangle = \langle 2|V|1\rangle \qquad \langle 2|V|2\rangle = \langle 1|V|1\rangle$$

The perturbation matrix is then

$$W = \beta a^3 \begin{pmatrix} \frac{1}{4} & \frac{256}{81\pi^4} \\ \frac{256}{81\pi^4} & \frac{1}{4} \end{pmatrix}$$

The eigenvalues of this matrix give first-order corrections to the energy of the fourth excited state

$$E^{(1)} = \beta a^3 \left( \frac{1}{4} \pm \frac{256}{81\pi^4} \right)$$

(3) In order to have discrete energy levels (bound states) the potential must be attractive. That is true only when $\alpha < 0$

To estimate the ground state energy the most obvious (and quite appropriate) trial wave function is the exponential one:

$$\psi(r) = e^{-\beta r} \qquad \beta \text{ is an adjustable parameter}$$

it has a proper asymptotic behavior when $r \to \infty$.

Let us now compute all necessary integrals

$$\langle\psi|\psi\rangle = 4\pi \int_0^\infty e^{-2\beta r} r^2 dr = \frac{\pi}{\beta^3}$$

$$\langle\psi|T|\psi\rangle = -\frac{\hbar^2}{2m} 4\pi \int_0^\infty e^{-\beta r}\left(\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} e^{-\beta r}\right) dr =$$

$$= -\frac{\hbar^2\pi}{m} \int_0^\infty (\beta r^2 - 2\beta r) e^{-2\beta r} dr = \frac{\pi\hbar^2}{2m\beta}$$

$$\langle\psi|V|\psi\rangle = -\alpha 4\pi \int_0^\infty e^{-\beta r} \frac{1}{\sqrt{r}} e^{-\beta r} r^2 dr = -4\pi\alpha \int_0^\infty r^{3/2} e^{-2\beta r} dr = -\frac{3\pi^{3/2}}{4\sqrt{2}} \frac{\alpha}{\beta^{5/2}}$$

(the last integral can be evaluated as $\int_0^\infty r^{3/2} e^{-\gamma r} dr \overset{y^2=r}{=} 2\int_0^\infty y^4 e^{-\gamma y^2} dy = 2\left(-\frac{\partial}{\partial\gamma}\right)^2 \sqrt{\frac{\pi}{\gamma}}$ )

Our trial energy is then

$$E = \frac{\langle\psi|T|\psi\rangle + \langle\psi|V|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\hbar^2}{2m}\beta^2 - \frac{3\pi^{1/2}}{4\sqrt{2}} \alpha \beta^{1/2}$$

Finding the minimum with respect to $\beta$ yields:

$$\frac{\partial E}{\partial\beta} = \frac{\hbar^2}{m}\beta - \frac{3\pi^{1/2}}{8\sqrt{2}} \alpha \frac{1}{\sqrt{\beta}} = 0 \implies \beta_{min} = \left(\frac{3\sqrt{\pi}\, m\alpha}{8\sqrt{2}\,\hbar^2}\right)^{2/3}$$

$$E_{min} = \frac{\hbar^2}{2m}\left(\frac{3\sqrt{\pi}\, m\alpha}{8\sqrt{2}\,\hbar^2}\right)^{4/3} - \frac{3\sqrt{\pi}}{4\sqrt{2}}\alpha\left(\frac{3\sqrt{\pi}\, m\alpha}{8\sqrt{2}\,\hbar^2}\right)^{1/3} = \frac{3^{4/3}\pi^{2/3}}{4\cdot 2^{11/3}} \frac{m^{1/3}\alpha^{4/3}}{\hbar^{2/3}} - \frac{3^{4/3}\pi^{2/3}}{2^{11/3}} \frac{m^{1/3}\alpha^{4/3}}{\hbar^{2/3}} =$$

$$= -\frac{3^{7/3}\pi^{2/3}}{2^{17/3}} \frac{m^{1/3}\alpha^{4/3}}{\hbar^{2/3}}$$

(4) First order corrections are given by

$$E_n^{(1)} = \langle \phi_n^{(0)} | H' | \phi_n^{(0)} \rangle = H'_{nn}$$

Second order corrections are given by

$$E_n^{(2)} = \sum_{k \neq n}' \frac{|H'_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} \qquad E_n^{(0)} = \hbar\omega\left(n+\tfrac{1}{2}\right)$$

where $\phi_n^{(0)}$ are the wave functions of the harmonic oscillator.
Matrix elements $H'_{nk} = \gamma \langle \phi_n^{(0)} | p^2 | \phi_k^{(0)} \rangle$ are given in the formula sheet (apart from multiplicative constant $\gamma$):

$$H'_{nk} = -\gamma \frac{m\hbar\omega}{2}\left[\sqrt{k(k-1)}\,\delta_{n,k-2} - (2k+1)\,\delta_{nk} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2}\right]$$

With that we have:

$$E_n^{(1)} = \gamma m\hbar\omega\left(n+\tfrac{1}{2}\right) = \gamma m E_n^{(0)}$$

(for the ground state we take $n=0$)

$$E_0^{(2)} = \sum_{k\neq 0} \frac{\gamma^2 \frac{m^2\hbar^2\omega^2}{4}\left[\sqrt{k(k-1)}\,\delta_{0,k-2} - (2k+1)\,\delta_{0,k} + \sqrt{(k+1)(k+2)}\,\delta_{0,k+2}\right]^2}{-\hbar\omega k} =$$

$$= -\frac{\gamma^2 m^2 \hbar\omega\,[\sqrt{2}]^2}{8} = -\frac{\gamma^2 m^2}{4}\hbar\omega$$