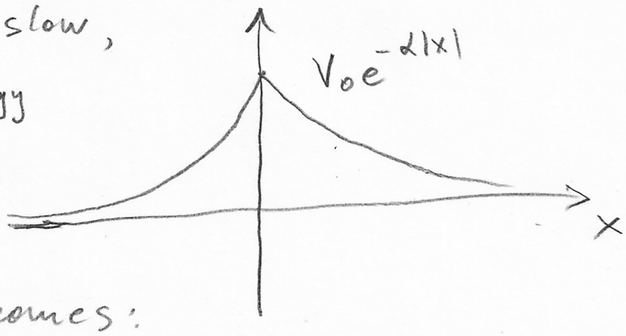
① According to the WKB approximation the tunneling probability is

$$T = e^{-2\gamma} \quad \text{with} \quad \gamma = \frac{\sqrt{2m}}{\hbar}\int_a^b \sqrt{V(x)-E}$$

here a and b are classical turning points.



Since our particle is very slow, we can assume that its energy approaches zero while the turning points approach $\pm\infty$. With that our γ becomes:

$$\gamma = \frac{\sqrt{2m}}{\hbar}\int_{-\infty}^{+\infty}\sqrt{V_0 e^{-\alpha|x|}} = \frac{2\sqrt{2mV_0}}{\hbar}\int_0^{\infty} e^{-\frac{\alpha|x|}{2}}dx = \frac{4\sqrt{2mV_0}}{\hbar\alpha}$$

and

$$T = e^{-\frac{8\sqrt{2mV_0}}{\hbar\alpha}}$$

The latter expression remains valid when $T \ll 1$. That gives us a constraint on α and V_0:

$$\frac{\sqrt{mV_0}}{\hbar\alpha} \gg 1$$

(2) To estimate the energy levels we can use the Bohr-Sommerfeld quantization rule for the case of a potential with one vertical wall:

$$\int_a^b p(x)\,dx = \left(n-\frac{1}{4}\right)\pi\hbar \qquad \text{where} \quad p(x)=\sqrt{2m(E-V(x))}$$

$n = 1, 2, 3, \ldots$

The turning points for our potential are

$$a = 0 \quad \text{and} \quad E - \frac{m\omega^2 b^2}{2} = 0 \Rightarrow b = \sqrt{\frac{2E}{m\omega^2}}$$

With that our quantization rule reads

$$\int_0^b \sqrt{2m\left(E-\frac{m\omega^2x^2}{2}\right)}\,dx = \left(n-\frac{1}{4}\right)\pi\hbar$$

or

$$m\omega\int_0^b \sqrt{\underbrace{\frac{2E}{m\omega^2}}_{b^2}-x^2}\,dx = \left(n-\frac{1}{4}\right)\pi\hbar$$

The integration yields (see formula sheet)

$$\frac{m\omega}{2}\left(x\sqrt{b^2-x^2} + b^2\arctan\left[\frac{x}{\sqrt{b^2-x^2}}\right]\right)\Big|_0^b = \left(n-\frac{1}{4}\right)\pi\hbar$$

$$\frac{m\omega b^2\pi}{4} = \left(n-\frac{1}{4}\right)\pi\hbar$$

$$\frac{m\omega}{4}\,\frac{2E}{m\omega^2} = \left(n-\frac{1}{4}\right)$$

$$E = \hbar\omega\left(2n-\frac{1}{2}\right) \qquad n = 1, 2, 3, \ldots$$

Hence the estimated energy levels are:

$$\frac{3}{2}\hbar\omega,\ \frac{7}{2}\hbar\omega,\ \frac{11}{2}\hbar\omega,\ \ldots$$

These happen to coincide with the exact eigenvalues (recall that because of the boundary condition $\psi(0)=0$ only odd states of the oscillator are valid solutions for "half-oscillator")

In general we would expect that

$$E_n^{WKB} \xrightarrow{n\to\infty} E_n^{exact}$$

3 a) $H^0 = \varepsilon I - \gamma B S_z = \varepsilon I - \frac{\gamma\hbar B}{2}\sigma_z$

The eigenvalues of σ_z are well known: ± 1. The corresponding normalized eigenvectors are

$$\chi_+ = \begin{pmatrix}1\\0\end{pmatrix} \qquad \chi_- = \begin{pmatrix}0\\1\end{pmatrix}$$

Adding the identity matrix just shifts the eigenvalues while the eigenvectors remain the same. So we have

$$E_+ = \varepsilon - \frac{\gamma\hbar}{2}B \qquad \chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$$

$$E_- = \varepsilon + \frac{\gamma\hbar}{2}B \qquad \chi_- = \begin{pmatrix}0\\1\end{pmatrix}$$

b) Here we can write the total time-dependent Hamiltonian as

$$H = H^0 + H' \quad \text{where} \quad H'(t) = -\frac{\gamma\hbar\beta}{2}(\sigma_x \cos\omega t - \sigma_y \sin\omega t) =$$

$$= -\frac{\gamma\hbar\beta}{2}\begin{pmatrix}0 & \cos\omega t + i\sin\omega t\\ \cos\omega t - i\sin\omega t & 0\end{pmatrix} = -\frac{\gamma\hbar\beta}{2}\begin{pmatrix}0 & e^{i\omega t}\\ e^{-i\omega t} & 0\end{pmatrix}$$

In the first order of the perturbation theory the probability amplitude $c_-^{(1)}$ is:

$$c_-^{(1)} = \frac{1}{i\hbar}\int_0^t \langle\chi_-|H'(t')|\chi_+\rangle e^{i\omega_{-+}t'}\,dt' \quad \text{where } \omega_{-+} = \frac{E_- - E_+}{\hbar} = \gamma B$$

$$\langle\chi_-|H'|\chi_+\rangle = -\frac{\gamma\hbar\beta}{2}(0\ \ 1)\begin{pmatrix}0 & e^{i\omega t}\\ e^{-i\omega t} & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = -\frac{\gamma\hbar B}{2}e^{-i\omega t}$$

With that we have

$$c_-^{(1)} = \frac{1}{i\hbar}\left(-\frac{\gamma\hbar\beta}{2}\right)\int_0^t e^{\underbrace{i(\gamma B-\omega)}_{\Omega = \gamma B - \omega}t'}\,dt' = \frac{\gamma\beta}{2}\,\frac{e^{i\Omega t}-1}{\Omega} = i\frac{\gamma\beta}{\Omega}e^{\frac{i\Omega t}{2}}\sin\frac{\Omega t}{2}$$

and the probability of transition from spin-up to spin-down is

$$P_{-\leftarrow+}^{(1)} = |c_-^{(1)}|^2 = \frac{\gamma^2\beta^2}{(\gamma B-\omega)^2}\sin^2\frac{\gamma B-\omega}{2}t$$

c) The perturbation theory remains valid when

$P_{-\leftarrow +} << 1$

This gives us a constraint on ω:

$|\gamma B - \omega| >> \gamma \beta$

(4) Let us assume that the first pulse begins at $t=0$ (the choice of t_0 is arbitrary as we need to see what happens when $t=+\infty$). The Hamiltonian of our system is

$$H = H^0 + \lambda x f(t) = H^0 + H'$$

where H^0 is the unperturbed Hamiltonian (harmonic oscillator) for which we know the solutions to the stationary Schrödinger equation ($E_n = \hbar\omega(n+\frac{1}{2})$, $|\phi_n\rangle$),

$$\lambda \equiv q\frac{U_{max}}{D},$$

and $f(t)$ is given by

$$f(t) = \begin{cases} 1, & 0<t<T \\ -1, & T<T<2T \\ 0, & \text{otherwise} \end{cases}$$

Since U_{max} is small so is λ. Therefore we can make use of the time-dependent perturbation theory. First-order probability amplitude is then (at any time $t>2T$):

$$c^{(1)}_{2\leftarrow 0} = \frac{1}{i\hbar}\int_0^{2T} \lambda\langle\phi_2|x|\phi_0\rangle f(t')\, e^{i\omega_{20}t'}\, dt'$$

Given that the value of the transition matrix element

$$\langle\phi_n|x|\phi_k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{k+1}\,\delta_{n,k+1}\right)$$

is nonzero only when n and k differ by ± 1 it is clear that

$$c^{(1)}_{2\leftarrow 0} = 0 \quad \text{and} \quad P^{(1)}_{2\leftarrow 0} = |c^{(1)}_{2\leftarrow 0}|^2 = 0$$

Hence we need to resort to the second order of the perturbation theory

$$c^{(2)}_{2\leftarrow 0} = \left(\frac{1}{i\hbar}\right)^2 \sum_{k=1}^{\infty} \int_0^{2T} dt' \int_0^{t'} \lambda^2 \langle\phi_2|x|\phi_k\rangle\langle\phi_k|x|\phi_0\rangle f(t')f(t'')\, e^{i\omega_{2k}t'}\, e^{i\omega_{k0}t''}\, dt'dt''$$

Here, again most transition matrix elements vanish. Only the term $k=1$ survives in the sum:

$$c^{(2)}_{2\leftarrow 0} = -\frac{\lambda^2}{\hbar^2}\langle\phi_2|x|\phi_1\rangle\langle\phi_1|x|\phi_0\rangle \int_0^{2T} dt' f(t')\, e^{i\omega t'} \int_0^{t'} dt'' f(t'')\, e^{i\omega t''}\, dt''$$

The second integral is

$$\int_0^{t'} dt'' f(t'') e^{i\omega t''} dt'' = \begin{cases} \frac{1}{i\omega}\left(e^{i\omega t'}-1\right), & 0<t'<T \\ \frac{1}{i\omega}\left(e^{i\omega T}-1\right) - \frac{1}{i\omega}\left(e^{i\omega t'} - e^{i\omega T}\right), & T<t'<2T \end{cases} \equiv g(t')$$

The first integral is

$$\int_0^{2T} dt' f(t') e^{i\omega t'} g(t') = \frac{1}{i\omega}\left[\int_0^T dt'\left(e^{i\omega t'}-1\right)e^{i\omega t'} - \int_T^{2T} dt'\left(e^{i\omega T}-1\right)e^{i\omega t'} + \int_T^{2T} dt'\left(e^{i\omega t'}-e^{i\omega T}\right)e^{i\omega t'}\right] =$$

$$= \frac{1}{i\omega}\left[\frac{\left(e^{i\omega T}-1\right)^2}{2i\omega} - \frac{e^{i\omega T}\left(e^{i\omega T}-1\right)^2}{i\omega} + \frac{e^{2i\omega T}\left(e^{i\omega T}-1\right)^2}{2i\omega}\right] =$$

$$= -\frac{1}{2\omega^2}\left[1-2e^{i\omega T}+e^{2i\omega T}\right]\left(e^{i\omega T}-1\right)^2 = -\frac{1}{2\omega^2}\left(e^{i\omega T}-1\right)^4 = -\frac{8}{\omega^2}e^{2i\omega T}\left(\sin\frac{\omega T}{2}\right)^4$$

With that the probability of $2\leftarrow 0$ transition is

$$P^{(2)}_{2\leftarrow 0} = \left|c^{(2)}_{2\leftarrow 0}\right|^2 = \frac{\lambda^4}{\hbar^4}\frac{\hbar}{m\omega}\cdot\frac{\hbar}{2m\omega}\frac{64}{\omega^4}\left(\sin\frac{\omega T}{2}\right)^8 = \frac{32\lambda^4}{\hbar^2 m^2\omega^6}\left(\sin\frac{\omega T}{2}\right)^8$$

$$= \frac{32}{\hbar^2 m^2\omega^6}\left(\frac{qU_{max}}{D}\right)^4\left(\sin\frac{\omega T}{2}\right)^8$$