The (instantaneous) energy eigenvalues are:

$$\det\begin{pmatrix} \alpha - E & \alpha e^{-\beta t^2} \\ \alpha e^{-\beta t^2} & -\alpha - E \end{pmatrix} = 0 \quad \Rightarrow \quad E = \pm \alpha \sqrt{1 + e^{-2\beta t^2}}$$

The gap between the eigenvalues, which defines the time scale of the time-dependce in the wave function, is greater or equal to 2α, or by order of magnitude

$$\Delta E \sim \alpha$$

the corresponding time scale $T = \frac{\hbar}{\Delta E} \sim \frac{\hbar}{\alpha}$

Now for the adiabatic approximation to be valid we must require that the characteristic time scale (τ) of changes of the Hamiltonian is much less than T.

In our case $\tau \sim \frac{1}{\sqrt{\beta}}$ (width of the gaussian). Therefore,

$$\frac{1}{\sqrt{\beta}} \gg \frac{\hbar}{\alpha}$$

or

$$\frac{\alpha}{\sqrt{\beta}} \gg \hbar$$