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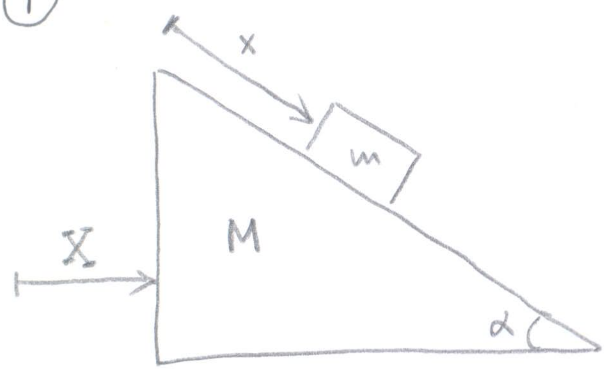


Let x be the distance of the block from the top of the wedge and X is the distance of the wedge from some point on the table.

Kinetic energy: $T = T_M + T_m = \frac{M}{2}\dot{X}^2 + \frac{m}{2}\left[(\dot{X} + \dot{x}\cos\alpha)^2 + (\dot{x}\sin\alpha)^2\right]$

$= \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}m\dot{x}^2 + m\dot{X}\dot{x}\cos\alpha$

Potential energy: $V = -mgx\sin\alpha$

Lagrangian: $L = \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}m\dot{x}^2 + m\dot{X}\dot{x}\cos\alpha + mgx\sin\alpha$

Lagrange equations are as follows

X: $(M+m)\ddot{X} + m\ddot{x}\cos\alpha = 0$

x: $m\ddot{x} + m\ddot{X}\cos\alpha = mg\sin\alpha$

From the second equation we get $\ddot{X} = \frac{1}{\cos\alpha}(g\sin\alpha - \ddot{x})$ which we can substitute into the first equation and obtain

$$(M+m)g\frac{\sin\alpha}{\cos\alpha} = \left(\frac{M+m}{\cos\alpha} - m\cos\alpha\right)\ddot{x}$$

or

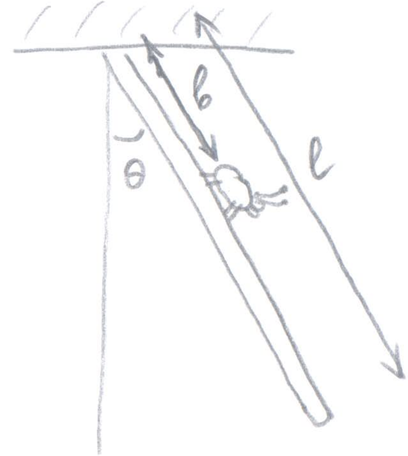
$$\ddot{x} = \frac{g\sin\alpha}{1 - \frac{m}{m+M}\cos^2\alpha}$$

Acceleration $\ddot{x}$ is constant. Hence the time it takes to slide distance ℓ is obtained from $\ell = \frac{\ddot{x}t^2}{2}$, so

$$t = \sqrt{\frac{2\ell}{\ddot{x}}} = \sqrt{\frac{2\ell\left(1 - \frac{m}{m+M}\cos^2\alpha\right)}{g\sin\alpha}}$$

② a) After the bug has crawled distance b, the Lagrangian of the system is

$$L = \frac{I\dot{\theta}^2}{2} + \frac{1}{2}\frac{M}{3}u^2 + \underbrace{\frac{1}{2}Mgl\cos\theta}_{-V_{rod}} + \underbrace{\frac{1}{3}Mgb\cos\theta}_{-V_{bug}}$$



Here I is the moment of inertia of the system that consists of the uniform rod and the bug: $I = \frac{1}{3}Ml^2 + \frac{1}{3}Mb^2 = \frac{1}{3}M(l^2+b^2)$

The Lagrange equation of motion, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$, becomes

$$\frac{d}{dt}\left(I\dot{\theta}\right) = -\frac{1}{2}Mgl\sin\theta - \frac{1}{3}Mgb\sin\theta$$

or

$$I\ddot{\theta} + \dot{I}\dot{\theta} = -Mg\left(\frac{l}{2}+\frac{b}{3}\right)\sin\theta$$

Now $\dot{I} = \frac{2}{3}Mb\dot{b} = \frac{2}{3}Mbu$. Thus,

$$\ddot{\theta} + \frac{2bu}{l^2+b^2}\dot{\theta} + \frac{\frac{3}{2}l+b}{l^2+b^2}g\sin\theta = 0$$ ← the equation of motion

b) Assuming u to be small we can neglect the second term in the equation of motion. We can also replace $\sin\theta$ with θ for small oscillations. Then we get

$$\ddot{\theta} + \underbrace{\frac{\frac{3}{2}l+b}{l^2+b^2}g}_{\omega^2}\theta = 0$$

So the frequency of small oscillations is

$$\omega = \sqrt{\frac{\frac{3}{2}l+b}{l^2+b^2}g}$$

③ The kinetic energy of the disk as it falls is

$T = \frac{1}{2} m\dot{y}^2 + \frac{1}{2} I \dot{\phi}^2$ where m is the mass of the disk and I is the moment of inertia about its center of mass, while y denotes the vertical position of the center of mass. ϕ is the angle of rotation about the center of mass. We know that for a uniform disk $I = \frac{ma^2}{2}$. The potential energy (assuming the y-axis point down) is given by $V = -mgy$. With all that our Lagrangian is

$$L = \frac{1}{2} m\dot{y}^2 + \frac{1}{4} ma^2\dot{\phi}^2 + mgy$$

Since the vertical position of the center of mass is related to the rotation angle as $y = a\phi$, the constraint can be written as

$$g(y,\phi) = y - a\phi = 0 \quad \text{(do not confuse } g(y,\phi) \text{ and } g\text{)}$$

For a system with a constraint the Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \lambda \frac{\partial g(y,\phi)}{\partial q_k}$$

In our case we get

$$y: \quad m\ddot{y} - mg = \lambda$$

$$\phi: \quad \frac{1}{2} ma^2 \ddot{\phi} = -\lambda a$$

These two equations and the equation of constraint can be solved easily. If we differentiate the equatioun of constraint we get $\ddot{\phi} = \frac{\ddot{y}}{a}$, which can be inserted into the equation for ϕ. That gives

$$\begin{cases} m\ddot{y} - mg = \lambda \\ \frac{1}{2} m\ddot{y} = -\lambda \end{cases} \Longrightarrow \lambda = -\frac{1}{3}mg \quad \ddot{y} = \frac{2}{3}g \quad \ddot{\phi} = \frac{2}{3}\frac{g}{a}$$

The forces of constraint are

$$Q_y = \lambda \frac{\partial g(y,\phi)}{\partial y} = \lambda = -\frac{1}{3}mg$$

$$Q_\phi = \lambda \frac{\partial g(y,\phi)}{\partial \phi} = -\lambda a = \frac{1}{3}mga$$

(4) For the circular orbits of radius R_1 and R_2 the velocity can be easily found by equating the gravitational and centrifugal forces

$$m\frac{v_1^2}{2} = G\frac{mM}{R_1^2}$$

$$m\frac{v_2^2}{2} = G\frac{mM}{R_2^2}$$

here m and M are the masses of the spacecraft and the Earth respectively

which yields $v_1 = \sqrt{\frac{GM}{R_1}}$ and $v_2 = \sqrt{\frac{GM}{R_2}}$. This immediately gives the asnswer to the last question – by what factor does the spacecraft velocity change in the whole maneuver:

$$\frac{v_2}{v_1} = \sqrt{\frac{R_1}{R_2}}$$

Now, the intermediate orbit is an elliptic one. At points P_1 and P_2 (perihelion and aphelion) the radial component of the velocity is zero. Let us denote v_p and v_a the tangential components at the perihelion and aphelion respectively. The conservation of the angular momentum, $l = mrv_t$, and the total energy, $E = \frac{mv^2}{2} + V(r)$ gives

$$\begin{cases} mR_1 v_p = mR_2 v_a \\ \frac{mv_p^2}{2} - G\frac{mM}{R_1} = \frac{mv_a^2}{2} - G\frac{mM}{R_2} \end{cases} \quad \text{or} \quad \begin{cases} R_1 v_p = R_2 v_a \\ v_p^2 - v_a^2 = GM\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \end{cases}$$

Solving for v_p and v_a yields

$$v_p = \sqrt{2GM\frac{R_2}{R_1}\frac{1}{R_1+R_2}} \qquad v_a = \sqrt{2GM\frac{R_1}{R_2}\frac{1}{R_1+R_2}}$$

Then we can determine the required thrust factors at points P_1 and P_2:

$$\lambda_1 = \frac{v_p}{v_1} = \sqrt{\frac{2R_2}{R_1+R_2}} \qquad \lambda_2 = \frac{v_a}{v_2} = \sqrt{\frac{2R_1}{R_1+R_2}}$$

(5) If the angular positions of the beads are denoted θ_1 and θ_2 respectively then the kinetic energy of the system is

$$T = \frac{1}{2} m R^2 \dot{\theta}_1^2 + \frac{1}{2} m R^2 \theta_2^2$$

The potential energy, on the other hand, is

$$V = \underbrace{\frac{1}{2} k R^2 (\theta_2 - \theta_1 - \pi)^2}_{\text{1st spring}} + \underbrace{\frac{1}{2} k R^2 (\theta_1 - \theta_2 - \pi)^2}_{\text{2nd spring}}$$

The Lagrangian is then

$$L = \frac{1}{2} m R^2 \dot{\theta}_1^2 + \frac{1}{2} m R^2 \dot{\theta}_2^2 - k R^2 (\theta_2 - \theta_1)^2 + \text{const}$$

The equations of motion are:

$$\theta_1 : \quad m R^2 \ddot{\theta}_1 - 2 k R^2 (\theta_2 - \theta_1) = 0$$

$$\theta_2 : \quad m R^2 \ddot{\theta}_2 + 2 k R^2 (\theta_2 - \theta_1) = 0$$

or

$$\begin{cases} \ddot{\theta}_1 + 2\omega_0^2 \theta_1 - 2\omega_0^2 \theta_2 = 0 \\ \ddot{\theta}_2 + 2\omega_0^2 \theta_2 - 2\omega_0^2 \theta_1 = 0 \end{cases}$$

In the matrix form this system of equations looks as follow

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{M} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = - \underbrace{\begin{pmatrix} 2\omega_0^2 & -2\omega_0^2 \\ -2\omega_0^2 & 2\omega_0^2 \end{pmatrix}}_{K} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

If we seek for the solution in the form

$$\vec{\theta} = \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t} = \vec{a} e^{i\omega t}$$

then we obtain an eigenvalue problem

$$K\vec{a} = \omega^2 M \vec{a}$$

which has a nontrivial solution when $\det(K - \omega^2 M) = 0$

Thus,

$$\begin{vmatrix} 2\omega_0^2 - \omega^2 & -2\omega_0^2 \\ -2\omega_0 & 2\omega_0^2 - \omega^2 \end{vmatrix} = 0$$

if $\lambda \equiv \frac{\omega^2}{\omega_0^2}$ then we get an algebraic equation

$$(2-\lambda)^2 - 4 = 0 \qquad \lambda_{1,2} = 4, 0$$

Hence the roots are $\omega_{1,2}^2 = 4\omega_0^2, 0$

The first root, $\omega_1^2 = 4\omega_0^2$ yields the following eigenvector

$$\begin{pmatrix} -2\omega_0^2 & -2\omega_0^2 \\ -2\omega_0^2 & -2\omega_0^2 \end{pmatrix} \begin{pmatrix} a_1^{(1)} \\ a_2^{(2)} \end{pmatrix} = 0 \qquad a_2^{(1)} = -a_2^{(1)} \qquad \vec{a}^{(1)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

So $\vec{\theta}^{(1)}(t) = A \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} e^{2i\omega_0 t}$ where A is a constant

The physically meaningful solution is, of course, obtained by taking the real part of $\vec{\theta}^{(1)}(t)$

The second root, $\omega_2^2 = 0$, needs to be treated with a little more care. Obviously

$$\begin{pmatrix} 2\omega_0^2 & -2\omega_0^2 \\ -2\omega_0^2 & 2\omega_0^2 \end{pmatrix} \begin{pmatrix} a_1^{(2)} \\ a_2^{(2)} \end{pmatrix} = 0 \implies a_2^{(2)} = a_1^{(2)} \qquad \vec{a}^{(2)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

The zero eigenvalue, $\omega_2^2 = 0$, does not correspond to the oscillatory motion. It correspond to the equation

$$\ddot{\xi}_2 = 0$$

in normal coordinates $\vec{\xi}$, where $\xi_2 = \frac{1}{\sqrt{2}}\theta_1 + \frac{1}{\sqrt{2}}\theta_2$. The solution of that equation

$$\xi_2(t) = \frac{1}{\sqrt{2}}\theta_1(t) + \frac{1}{\sqrt{2}}\theta_2(t) = \alpha t + \beta \qquad \alpha, \beta = \text{const}$$

It corresponds to a motion where both beads rotate with a constant angular velocity

The general solution of the equation of motion is

$$\vec{\theta}(t) = \vec{\theta}^{(1)}(t) + \vec{\theta}^{(2)}(t) = A \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} e^{2i\omega_0 t} + (Bt + C) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

where A is a complex constant and B and C are real constants. (effectively there are four real constants)

⑥ a) Given the Lagrangian $L = f(t)\left[\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\right]$ the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

or

$$\frac{d}{dt}\left(f(t)\, m\dot{q}\right) - f(t)\, m\omega^2 q = 0$$

or

$$f\ddot{q} + \dot{f}\dot{q} - f\omega^2 q = 0$$

or

$$\ddot{q} + \frac{\dot{f}}{f}\dot{q} - \omega^2 q = 0$$

Hence we see that $\frac{\dot{f}}{f} = 2\gamma$ and $f(t) = e^{2\gamma t}$ $(f(0) = 1)$

b) $p \equiv \frac{\partial L}{\partial \dot{q}} = f(t)\, m\dot{q} \Rightarrow \dot{q} = \frac{p}{mf}$

Then the Hamiltonian is

$$H = p\dot{q} - L(q, \dot{q}(p,q), t) = \frac{p^2}{mf} - f\left[\frac{1}{2}m\frac{p^2}{m^2 f^2} - \frac{1}{2}m\omega^2 q^2\right] =$$

$$= \frac{p^2}{2m}e^{-2\gamma t} + \frac{m\omega^2 q^2}{2}e^{2\gamma t}$$

c) The original and transformed Hamiltonians are related via

$$p\dot{q} - H = P\dot{Q} - H' + \frac{d}{dt}\left(F_2(q,P,t) - QP\right)$$

or, if we take the time derivative,

$$p\dot{q} - H = \cancel{P\dot{Q}} - H' + \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q}\dot{q} + \frac{\partial F_2}{\partial P}\dot{P} - \cancel{\dot{Q}P} - Q\dot{P}$$

from which it follows that

$$p = \frac{\partial F_2}{\partial q} \qquad Q = \frac{\partial F_2}{\partial P} \qquad H' = H + \frac{\partial F_2}{\partial t}$$

So when $F_2 = e^{\gamma t} qP$ we obtain

$$p = \frac{\partial F_2}{\partial q} = e^{\gamma t}P \qquad Q = \frac{\partial F_2}{\partial P} = e^{\gamma t}q \Rightarrow q = e^{-\gamma t}Q$$

$$H' = \frac{e^{2\gamma t}P^2}{2m}e^{-2\gamma t} + \frac{m\omega^2 e^{-2\gamma t}Q^2}{2}e^{2\gamma t} + \gamma e^{\gamma t}e^{-\gamma t}QP = \frac{P^2}{2m} + \frac{m\omega^2 Q^2}{2} + \gamma QP$$

H' doest not have any explicit time dependence and is, therefore, conserved