1 $\nabla e^{-\alpha r} = -\alpha \frac{\vec r}{r} e^{-\alpha r}$ $\quad r \equiv |\vec r| \quad \nabla r = \frac{\vec r}{r} \quad \nabla \cdot \vec r = 3$

$$\nabla^2 e^{-\alpha r} = -\alpha\left(\frac{\vec r}{r}\cdot\left(-\alpha\frac{\vec r}{r}\right) - \frac{\vec r}{r^3}\cdot\vec r + 3\frac{1}{r}\right)e^{-\alpha r} = \left(\alpha^2 - \frac{2\alpha}{r}\right)e^{-\alpha r}$$

2 No, there is no such a field. The easiest way to demonstrate it is to recall a vector calculus identity $\nabla\cdot(\nabla\times\vec F) = 0$ for any $\vec F$.

In our case

$$\nabla\cdot(\nabla\times\vec F) = \frac{\partial(x+y)}{\partial x} + \frac{\partial(z)}{\partial y} + \frac{\partial(y^2)}{\partial z} = 1 \neq 0$$

Hence $\nabla\times\vec F$ cannot yield $(x+y, z, y^2)$

3 Here we need to compute the derivative of T along the direction of the bird's motion

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} = (\nabla T)\cdot\vec v$$

$$\nabla T = \left(2[10-x]e^{-(10-x)^2}, -2ye^{-y^2}, -2ze^{-z^2}\right)$$

$$\vec v = \frac{d\vec r}{dt} = \left(3t^2, 2\cos t, -e^{-t}-1\right)$$

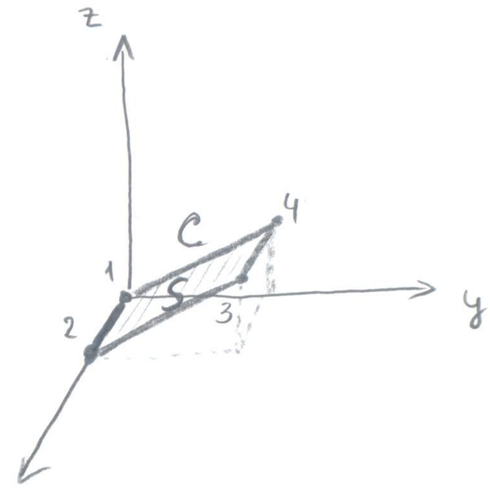
at $t=0$ we have:

$$\vec r = (0,0,1) \qquad \vec v = (0,2,-2)$$

then

$$\left.\frac{dT}{dt}\right|_{t=0} = 20e^{-10^2}\cdot 0 + 0\cdot 2 - 2e^{-1}(-2) = \frac{4}{e}\ \frac{\text{Kelvin}}{\text{second}}$$

④ a) The direct approach involves parametrizing the four line segments and computing the corresponding line integrals along those segments



Segment 1-2:

$d\vec{r} = \hat{x}\, d\lambda \qquad \vec{F} = (0,0,0) \qquad \lambda = [0,1]$, where λ is a parameter

$$W_{12} = \int_{C_{12}} \vec{F}\cdot d\vec{r} = 0$$

Segment 2-3:

$d\vec{r} = (2\hat{y} + \hat{z})\, d\lambda \qquad \vec{F} = (\lambda^2, 4\lambda, 16\lambda^2) \qquad \lambda = [0,1]$

$$W_{23} = \int_{C_{23}} \vec{F}\cdot d\vec{r} = \int_0^1 8\lambda\, d\lambda + \int_0^1 16\lambda^2 d\lambda = 4 + \frac{16}{3} = \frac{28}{3}$$

Segment 3-4:

$d\vec{r} = -\hat{x}\, d\lambda \qquad \vec{F} = (1, 2\lambda, 8) \qquad \lambda = [0,1]$

$$W_{34} = \int_{C_{34}} \vec{F}\cdot d\vec{r} = -\int_0^1 d\lambda = -1$$

Segment 4-1:

$d\vec{r} = -(2\hat{y} + \hat{z})\, d\lambda \qquad \vec{F} = \left((1-\lambda)^2, 0, 16(1-\lambda)^2\right)$

$$W_{41} = \int_{C_{41}} \vec{F}\cdot d\vec{r} = -\int_0^1 16(1-\lambda)^2 d\lambda = -\frac{16}{3}$$

Total work is then $W = W_{12} + W_{23} + W_{34} + W_{41} = 0 + \frac{28}{3} - 1 - \frac{16}{3} = 3$

b) According to the Stokes theorem

$$\int_C \vec{F}\cdot d\vec{r} = \int_S (\nabla\times\vec{F})\cdot\vec{n}\, ds$$

where S is the surface enclosed by loop C

if $\vec{F} = (z^2, 2xy, 4y^2)$ then $\nabla\times\vec{F} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & 2xy & 4y^2 \end{pmatrix} = 8y\hat{x} + 2z\hat{y} + 2y\hat{z}$

The normal unit vector is $\hat{n} = \left(0, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

$\vec{n}\, dS = \vec{n} \cdot \frac{\sqrt{5}}{2}\, dx\, dy \qquad 0 \le x \le 1 \quad 0 \le y \le 2$

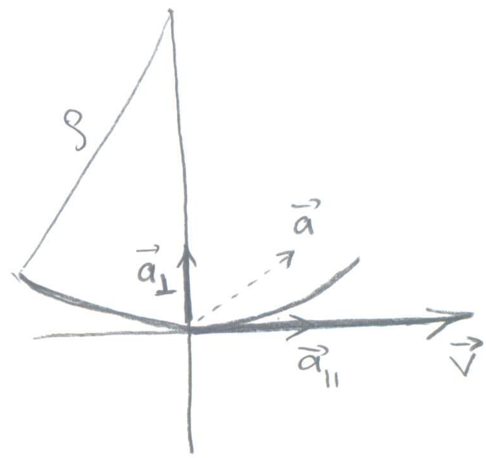
$z = \frac{1}{2} y$

Thus,

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 dx \int_0^2 dy \left[(8y, 2z, 2y) \cdot \left(0, -\tfrac{1}{2}, 1\right) \right] =$$

$$= \int_0^1 dx \int_0^2 dy \left[-z + 2y \right] = \frac{3}{2} \int_0^1 dx \int_0^2 y\, dy = \frac{3}{2} \cdot 1 \cdot \frac{4}{2} = 3$$

5 When we compute $\vec{v} \times \vec{a}$ only the perpendicular (to the instantaneous velocity) component of the acceleration contributes to the vector product. For any short time interval dt, as illustrated in the sketch, the situation is essentially identical to the circular motion of a particle, for which we know that



$$a_\perp = \frac{v^2}{\rho}$$

Using this fact we easily obtain

$$|\vec{v} \times \vec{a}| = v\, a_\perp = \frac{v^3}{\rho}$$

⑥ Second Newton's law : $\sum \vec{F} = m\vec{a}$

In our case it reads

$$mg - \beta v = ma$$

or

$$\dot{v} = g - \frac{\beta}{m} v$$

We can integrate this equation easily

$$\int \frac{dv}{g - \frac{\beta}{m} v} = \int dt$$

$$-\frac{m}{\beta} \ln\left(v - \frac{mg}{\beta}\right) + C = t \qquad C \text{ is a constant}$$

from which we solve for v:

$$v = \frac{mg}{\beta} + C' e^{-\frac{\beta}{m} t} \qquad C' \text{ is another constant related to } C$$

Since $v(0) = 0$ we obtain that $C' = -\frac{mg}{\beta}$ and

$$v(t) = \frac{mg}{\beta}\left(1 - e^{-\frac{\beta t}{m}}\right)$$

The speed with which the ball hits the Earth corresponds to the value of $v(t)$ at large t:

$$v(\infty) = \frac{mg}{\beta}$$

⑦ The gravitational force on the surface of the Earth is $F = m\frac{GM_{Earth}}{R^2_{Earth}} = mg$. As it is known (I won't show it here but it can be derived easily) for a point object of mass m that interacts with a sphere of a uniform density the gravitational force is

$$F(r) = \begin{cases} -m\frac{GM}{R^2} & r > R \\ -mG\frac{M(r)}{r^2} & r < R \end{cases}$$

where $M(r)$ is the mass of a sphere of radius r. Obviously $M(r) = \frac{4}{3}\pi r^3 \rho$ where ρ is the density of the Earth (we assume it is uniform)

Therefore we have the following equation of motion

$$m\ddot{r} = -mG\frac{4}{3}\pi\rho r \quad \text{or} \quad \ddot{r} + \underbrace{\frac{4}{3}\pi G\rho}_{\omega^2} r = 0$$

This is a harmonic oscillator equation. The period is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{1}{\frac{4}{3}\pi G\rho}} = \sqrt{\frac{3\pi}{G\rho}}$. The travel time from surface to surface is half the period, $\tau = \frac{T}{2}$. We can express these in terms of the radius of the Earth and g (at the surface) rather than ρ and G:

$$G = \frac{R^2 g}{M} \qquad G\cdot\rho = \frac{R^2 g}{M}\cdot\frac{M}{\frac{4}{3}\pi R^3} = \frac{3g}{4\pi R}$$

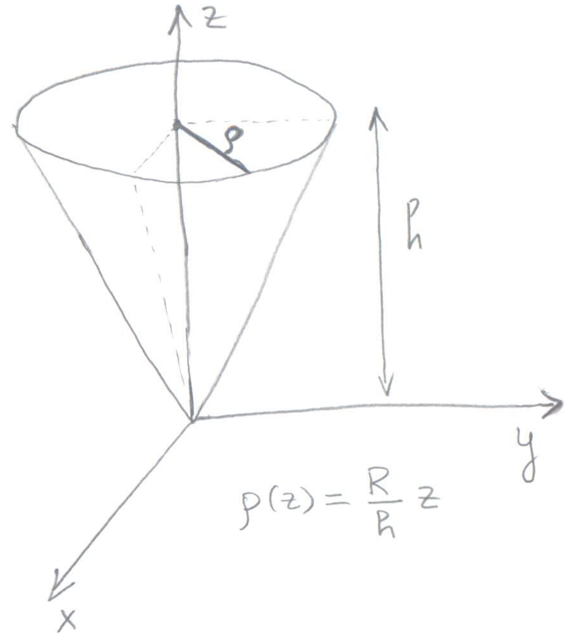
$$T = 2\pi\sqrt{\frac{R}{g}} \qquad \tau = \pi\sqrt{\frac{R}{g}}$$

$$T \approx 5068 \text{ sec.}$$

(8) In this problem we assume the cone of a uniform density σ

$$\sigma = \frac{m}{V} = \frac{m}{\frac{1}{3}\pi R^2 h}$$

First, let us find the position of the center of mass of the cone with the vertex placed at origin.



$$z_{CM} = \frac{\int_0^h dz \int_0^{\rho(z)} \rho\, d\rho \int_0^{2\pi} d\phi\, \sigma \cdot z}{\int_0^h dz \int_0^{\rho(z)} \rho\, d\rho \int_0^{2\pi} d\phi\, \sigma} =$$

$$= \frac{2\pi\sigma \int_0^h z\, dz \int_0^{Rz/h} \rho\, d\rho}{2\pi\sigma \int_0^h dz \int_0^{Rz/h} \rho\, d\rho} = \frac{\int_0^h z \frac{\left(\frac{Rz}{h}\right)^2}{2} dz}{\int_0^h \frac{\left(\frac{Rz}{h}\right)^2}{2} dz} =$$

$$= \frac{\int_0^h z^3 dz}{\int_0^h z^2 dz} = \frac{3}{4} h$$

Second, let us find the moment of inertia of the cone about the x-axis (or any other axis lying in the xy-plane)

$$I_x = \int_0^h dz \int_0^{\rho(z)} d\rho \int_0^{2\pi} d\phi\, (z^2 + y^2)\, \sigma =$$

$$= \sigma \underbrace{\int_0^h dz \int_0^{Rz/h} d\rho \int_0^{2\pi} d\phi\, (z^2 + \rho^2 \sin^2\phi)}_{\frac{1}{5}\pi h^3 R^2 + \frac{1}{20}\pi h R^4} = \frac{m}{\frac{1}{3}\pi R^2 h}\left(\frac{1}{5}\pi h^3 R^2 + \frac{1}{20}\pi h R^4\right)$$

$$= m\left(\frac{3}{5} h^2 + \frac{3}{20} R^2\right)$$

Now if we shift the cone along the z-axis by ℓ then its moment of inertia can be determined by the parallel axis theorem:

$$I' = I_0 + ml^2 = m\left(l^2 + \frac{3}{5}h^2 + \frac{3}{20}R^2\right)$$

Now we have a physical pendulum with an effective radius of oscillation

$$L_{eff} = \frac{I}{m\,L_{CM}} = \frac{m\left(l^2 + \frac{3}{5}h^2 + \frac{3}{20}R^2\right)}{m\left(l + \frac{3}{4}h\right)}$$

The period of oscillation is given by

$$T = 2\pi\sqrt{\frac{L_{eff}}{g}} = 2\pi\sqrt{\frac{l^2 + \frac{3}{5}h^2 + \frac{3}{20}R^2}{g\left(l + \frac{3}{4}h\right)}}$$

(9) Goldstein 1.1

$$T = \frac{\vec{p}^2}{2m} \qquad \frac{dT}{dt} = \frac{d}{dt}\frac{\vec{p}^2}{2m} = \frac{2\vec{p}\cdot\frac{d\vec{p}}{dt}}{2m} = \vec{v}\cdot\vec{F}$$

if $m \neq$ const then

$$\frac{d\,mT}{dt} = \frac{d}{dt}\left(\frac{\vec{p}^2}{2}\right) = \frac{2\vec{p}\cdot\frac{d\vec{p}}{dt}}{2} = \vec{p}\cdot\vec{F}$$

(10) Goldstein 1.2

Consider a system of just two particles. The laws of their motion are

$$\dot{\vec{p}}_1 = \vec{F}_{12} + \vec{F}_1^{ext}$$

$$\dot{\vec{p}}_2 = \vec{F}_{21} + \vec{F}_2^{ext}$$

if we add the two equations above we get

$$\underbrace{\dot{\vec{p}}_1 + \dot{\vec{p}}_2}_{\vec{P}} = \vec{F}_{12} + \vec{F}_{21} + \underbrace{\vec{F}_1^{ext} + \vec{F}_2^{ext}}_{\vec{F}^{ext}}$$

since $\vec{P} = M\frac{d^2\vec{R}}{dt^2} = \vec{F}^{ext}$ (equation 1.22) it must be that

$$\vec{F}_{12} + \vec{F}_{21} = 0 \quad \leftarrow \text{weak law of action and reaction}$$

Now consider the torques in the same system of two particles:

$$\vec{L} = \vec{L}_1 + \vec{L}_2 = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

Les us take the time derivative:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}_1 \times \vec{p}_1) + \frac{d}{dt}(\vec{r}_2 \times \vec{p}_2)$$

$$\frac{d\vec{L}}{dt} = \underbrace{\frac{d\vec{r}_1}{dt}\times\vec{p}_1}_{0} + \vec{r}_1 \times \frac{d\vec{p}_1}{dt} + \underbrace{\frac{d\vec{r}_2}{dt}\times\vec{p}_2}_{0} + \vec{r}_2 \times \frac{d\vec{p}_2}{dt}$$

since $\frac{d\vec{r}}{dt} \parallel \vec{p}$

$$\frac{d\vec{L}}{dt} = \vec{r}_1 \times \left(\vec{F}_{12} + \vec{F}_1^{ext}\right) + \vec{r}_2 \times \left(\vec{F}_{21} + \vec{F}_2^{ext}\right)$$

$$= \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21} + \vec{N}^{ext}$$

According to equation 1.26, $\frac{d\vec{L}}{dt} = \vec{N}^{ext}$. Therefore, the term

$$\vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21} = \left(\vec{r}_1 - \vec{r}_2\right) \times \vec{F}_{12} = \vec{r}_{12} \times \vec{F}_{12}$$

must vanish. This is only possible if $\vec{F}_{12} \parallel \vec{r}_{12}$, which is essentially gives the strong law of action and reaction.