(1) First let us find how the mass of the raindrop changes with time:

$$\frac{dm}{dt} = \alpha S$$ where S is the surface are and α is a proportionality constant (we assume α is known).

$m = \rho V$, where ρ is the density of water. Hence, $r = \left(\frac{3m}{4\pi\rho}\right)^{1/3}$

$$S = 4\pi r^2 = 4\pi\left(\frac{3m}{4\pi\rho}\right)^{2/3} = \frac{(36\pi)^{1/3}}{\rho^{2/3}} m^{2/3}$$

With that we get the equation

$$\frac{dm}{dt} = \frac{\alpha(36\pi)^{1/3}}{\rho^{2/3}} m^{2/3}$$

$$\int \frac{dm}{m^{2/3}} = \frac{\alpha(36\pi)^{1/3}}{\rho^{2/3}} \int dt$$

$$3m = \frac{\alpha(36\pi)^{1/3}}{\rho^{2/3}} t + C'$$

$$m(t) = \left(\frac{\alpha(36\pi)^{1/3}}{9\rho^{2/3}} t + C\right)^3$$

Constant C can be found from the initial condition $m(0) = \frac{4}{3}\pi\rho r_0^3$

$$\frac{4}{3}\pi\rho r_0^3 = C^3 \qquad C = r_0\left(\frac{4}{3}\pi\rho\right)^{1/3}$$

Then

$$m(t) = \left(\frac{\alpha\left(\frac{4}{3}\pi\right)^{1/3}}{\rho^{2/3}} t + r_0\left(\frac{4}{3}\pi\rho\right)^{1/3}\right)^3 = \frac{4}{3}\pi\rho\left(r_0 + \frac{\alpha t}{\rho}\right)^3$$

Now let us consider the motion of a variable-mass raindrop in a uniform gravitational field

$$\frac{dp}{dt} = F \qquad \frac{d(mv)}{dt} = -mg \qquad m\frac{dv}{dt} + v\frac{dm}{dt} = -mg$$

$$\frac{dv}{dt} = -\left(\frac{1}{m}\frac{dm}{dt}\right)v - g \qquad \frac{1}{m}\frac{dm}{dt} = \frac{4\pi\rho\left(r_0+\frac{\alpha t}{\rho}\right)^2\frac{\alpha}{\rho}}{\frac{4}{3}\pi\rho\left(r_0+\frac{\alpha t}{\rho}\right)^3} = \frac{3\alpha}{\rho}\,\frac{1}{r_0+\frac{\alpha t}{\rho}} = \frac{3}{t+\frac{\rho r_0}{\alpha}}$$

Introducing the notation $\beta = \frac{\rho r_0}{\alpha}$ and $\tau = t + \beta$ we obtain

$$\frac{dv}{d\tau} = -\frac{3}{\tau}v - g$$

The solution of the above equation is

$$v(t) = -\frac{g}{4}(t+\beta) + \frac{C}{(t+\beta)^3}$$

Constant C is found from the initial condition $v(0)=0$

$$0 = v(0) = \frac{C}{\beta^3} - \frac{g\beta}{4} \quad \Rightarrow \quad C = \frac{g\beta^4}{4}$$

and

$$v(t) = \frac{g}{4}\left(\frac{\beta^4}{(t+\beta)^3} - (t+\beta)\right)$$

Then we can easily integrate it to get $y(t)$

$$y(t) = C + \int v(t)\,dt = C - \frac{gt^2}{8} - \frac{g\beta t}{4} - \frac{g\beta^4}{8(t+\beta)^2} \qquad C \equiv \text{const}$$

Constant C is found from the condition $y(0) = y_0$

$$y_0 = C - \frac{g\beta^4}{8\beta^2} = C - \frac{g\beta^2}{8} \quad \Rightarrow \quad C = y_0 + \frac{g\beta^2}{8}$$

Then

$$y(t) = y_0 + \frac{g\beta^2}{8} - \frac{gt^2}{8} - \frac{g\beta t}{4} - \frac{g\beta^4}{8(t+\beta)^2}$$

The time it takes to reach the surface is found by solving the equation

$$y_0 + \frac{g\beta^2}{8} - \frac{gt^2}{8} - \frac{g\beta t}{4} - \frac{g\beta^4}{8(t+\beta)^2}$$

This equation is equivalent to a cubic one. The solution is

$$t = \beta\left(-1 + \sqrt{1 + 4\frac{y_0}{\beta^2 g} + \frac{2\sqrt{2}\,y_0}{\beta^2 g}\sqrt{2 + \frac{\beta^2 g}{y_0}}}\right) = \beta\left(-1 + \sqrt{1 + 4\gamma + 2\sqrt{2}\,\gamma\sqrt{2 + \frac{1}{\gamma}}}\right)$$

where $\gamma = \frac{y_0}{\beta^2 g} = \frac{y_0 d^2}{\rho^2 r_0^2 g}$

(2) a) $T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$x = \ell \sin\phi \sin\theta \qquad \dot{x} = \ell(\cos\phi \sin\theta\, \dot{\phi} + \sin\phi \cos\theta\, \dot{\theta})$

$y = \ell \cos\phi \sin\theta \qquad \dot{y} = \ell(-\sin\phi \sin\theta\, \dot{\phi} + \cos\phi \cos\theta\, \dot{\theta})$

$z = \ell \cos\theta \qquad \dot{z} = -\ell \sin\theta\, \dot{\theta}$

$$T = \frac{m\ell^2}{2}\Big(\cos^2\phi \sin^2\theta\, \dot{\phi}^2 + \sin^2\phi \cos^2\theta\, \dot{\theta}^2 + \underline{2\cos\phi \sin\phi \sin\theta \cos\theta\, \dot{\phi}\dot{\theta}}$$
$$+ \sin^2\phi \sin^2\theta\, \dot{\phi}^2 + \cos^2\phi \cos^2\theta\, \dot{\theta}^2 - \underline{2\sin\phi\cos\phi \sin\theta\cos\theta\, \dot{\phi}\dot{\theta}} + \sin^2\theta\, \dot{\theta}^2\Big)$$
$$= \frac{m\ell^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right)$$

$$V = -mg\ell\cos\theta$$

$$L = \frac{m\ell^2}{2}(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) - mg\ell\cos\theta$$

b) Lagrange equations:

$\theta: \quad \frac{d}{dt}(m\ell^2\dot{\theta}) = m\ell^2\dot{\phi}^2 \sin\theta\cos\theta - mg\ell\sin\theta$

$\phi: \quad \frac{d}{dt}(m\ell^2 \sin^2\theta\, \dot{\phi}) = 0$

or

$$\ddot{\theta} - \sin\theta\cos\theta\, \dot{\phi}^2 + \frac{g}{\ell}\sin\theta = 0$$

$$2\sin\theta\cos\theta\, \dot{\theta}\dot{\phi} + \sin^2\theta\, \ddot{\phi} = 0$$

c) $L_z = m\ell^2 \sin^2\theta_0\, \dot{\phi} = \text{const}$ where in our case $\theta_0 = \frac{\pi}{4}$, $\dot{\phi}^2 = \frac{g}{\ell\cos\theta_0}$ and $L_z^2 = \frac{m^2 g \ell^3 \sin^4\theta_0}{\cos\theta_0}$

or, alternatively, we can write

d) $\dot{\phi} = \frac{L_z}{m\ell^2\sin^2\theta}$ and we plug it into the equation for θ:

$$\ddot{\theta} - \frac{L_z^2 \cos\theta}{m^2\ell^4\sin^3\theta} + \frac{g}{\ell}\sin\theta = 0 \quad \text{or} \quad \ddot{\theta} - \frac{g}{\ell}\frac{\sin^4\theta_0}{\cos\theta_0}\frac{\cos\theta}{\sin^3\theta} + \frac{g}{\ell}\sin\theta = 0$$

e) When $\theta = \theta_0 + \xi$ and $\xi \to 0$ the last equation can be approximated using the Taylor expansions

$$\text{sen}(\theta_0+\xi) = \sin\theta_0 + \cos\theta_0\,\xi + O(\xi^2)$$

$$\cos(\theta_0+\xi) = \cos\theta_0 - \sin\theta_0\,\xi + O(\xi^2)$$

With that it becomes

$$\ddot{\xi} + \frac{g}{e}\left[\sin\theta_0 + (\cos\theta)\xi + \ldots\ldots - \right.$$

$$\left. - \frac{\sin^4\theta_0}{\cos\theta_0}\,\frac{\cos\theta_0 - (\sin\theta_0)\xi + \ldots\ldots}{(\sin\theta_0 + (\cos\theta_0)\xi + \ldots\ldots)^3}\right]$$

or

$$\ddot{\xi} + \frac{g}{e}\left[\sin\theta_0 + (\cos\theta_0)\xi - \sin\theta_0\left\{1 - \frac{\sin\theta_0}{\cos\theta_0}\xi + \ldots\right\}\cdot\left\{1 + \right.\right.$$

$$\left.\left. - \frac{\cos\theta_0}{\sin\theta_0}\xi + \ldots\right\}^3\right]$$

or

$$\ddot{\xi} + \frac{g}{e}\left[(\cos\theta_0)\xi + \frac{\sin^2\theta_0}{\cos\theta_0}\xi + 3\cos\theta_0\,\xi + \ldots\right] = 0$$

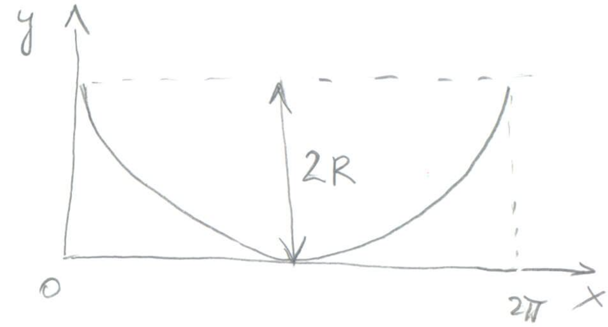
Keeping only the terms up to the first order in the square brackets yields

$$\ddot{\xi} + \frac{g}{e}\left(\frac{1}{\cos\theta_0} + 3\cos\theta_0\right)\xi = 0$$

This is a harmonic oscillator with angular frequency

$$\omega = \sqrt{\frac{g}{e}\left(\frac{1}{\cos\theta_0} + 3\cos\theta_0\right)} = \sqrt{\frac{5g}{\sqrt{2}e}}$$

③ $\begin{cases} x = R(\xi - \sin\xi) \\ y = R(1+\cos\xi) \end{cases}$ $0 \le \xi \le 2\pi$



a)

$\dot{x} = R(\dot{\xi} - \cos\xi\,\dot{\xi})$

$\dot{y} = -R\sin\xi\,\dot{\xi}$

Kinetic energy: $T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}mR^2\left[(1-\cos\xi)^2\dot{\xi}^2 + \sin^2\xi\,\dot{\xi}^2\right]$

$= mR^2\left[1-\cos\xi\right]\dot{\xi}^2$

Potential energy: $V = mgR(1+\cos\xi)$

The Lagrangian is $L = mR^2(1-\cos\xi)\dot{\xi}^2 - mgR(1+\cos\xi)$

b) $\frac{\partial L}{\partial\dot{\xi}} = 2mR^2(1-\cos\xi)\dot{\xi}$ $\qquad \frac{\partial L}{\partial \xi} = mR^2\sin\xi\,\dot{\xi}^2 + mgR\sin\xi$

The equation of motion:

$$\frac{d}{dt}\left(2mR^2(1-\cos\xi)\dot{\xi}\right) - mR^2\sin\xi\,\dot{\xi}^2 - mgR\sin\xi = 0$$

or

$$2(1-\cos\xi)\ddot{\xi} + 2\sin\xi\,\dot{\xi}^2 - \sin\xi\,\dot{\xi}^2 - \frac{g}{R}\sin\xi = 0$$

$$2(1-\cos\xi)\ddot{\xi} + \sin\xi\,\dot{\xi}^2 - \frac{g}{R}\sin\xi = 0$$

or $$\ddot{\xi} + \frac{1}{2}\frac{\sin\xi}{1-\cos\xi}\left(\dot{\xi}^2 - \frac{g}{R}\right) = 0$$

Here we can use the identity $\cot\frac{\xi}{2} = \frac{\sin\xi}{1-\cos\xi}$

With that we get

$$\ddot{\xi} + \frac{1}{2}\cot\frac{\xi}{2}\left(\dot{\xi}^2 - \frac{g}{R}\right) = 0$$

c) If we make a substitution $u = \cos\frac{\xi}{2}$ then

$\dot{u} = -\frac{1}{2}\left(\sin\frac{\xi}{2}\right)\dot{\xi}$ $\qquad \ddot{u} = -\frac{1}{2}\left(\sin\frac{\xi}{2}\right)\ddot{\xi} - \frac{1}{4}\left(\cos\frac{\xi}{2}\right)\dot{\xi}^2$

$\dot{\xi} = -\frac{2\dot{u}}{\sqrt{1-u^2}}$ $\qquad \ddot{\xi} = -\frac{2\ddot{u}}{\sqrt{1-u^2}} - \frac{2u\dot{u}^2}{(1-u^2)}$

This simplifies the equation to

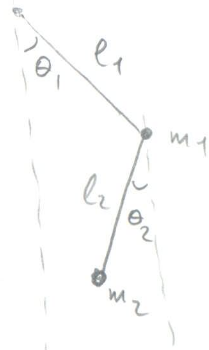
$$\ddot{u} + \frac{g}{4R} u = 0$$

The solution is

$$u = A \cos\left(\sqrt{\frac{g}{4R}}\, t + \alpha\right)$$ where A, α are constants

$$\xi = 2 \arccos\left[A \cos\left(\sqrt{\frac{g}{4R}}\, t + \alpha\right)\right]$$

④ Goldstein 1.22



$$x_1 = l_1 \sin\theta_1$$
$$y_1 = l_1 \cos\theta_1$$
$$x_2 = x_1 + l_2 \sin\theta_2$$
$$y_2 = y_1 + l_2 \cos\theta_2$$

$$T = \tfrac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2} m (\dot{x}_2^2 + \dot{y}_2^2)$$

$$V = - m_1 g y_1 - m_2 g y_2$$

$$\dot{x}_1 = l_1 \cos\theta_1 \, \dot{\theta}_1 \qquad \dot{x}_2 = \dot{x}_1 + l_2 \cos\theta_2 \, \dot{\theta}_2$$
$$\dot{y}_1 = -l_2 \sin\theta_1 \, \dot{\theta}_1 \qquad \dot{y}_2 = \dot{y}_1 - l_2 \sin\theta_2 \, \dot{\theta}_2$$

$$\dot{x}_2 + \dot{y}_2 = \dot{x}_1^2 + 2\dot{x}_1 l_2 \cos\theta_2 \, \dot{\theta}_2 + l_2^2 \cos^2\theta_2 \, \dot{\theta}_2^2 + \dot{y}_1^2 - 2\dot{y}_1 l_2 \sin\theta_2 \, \dot{\theta}_2 + l_2^2 \sin^2\theta_2 \, \dot{\theta}_2^2$$
$$= l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2$$

Therefore our Lagrangian is:

$$L = \frac{1}{2}(m_1 l_1^2 + m_2 l_2^2)\dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 +$$
$$+ (m_1 g + m_2 g) l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

Partial derivatives are

$$\frac{\partial L}{\partial \theta_1} = - m_2 l_1 l_2 \sin(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 - (m_1 g + m_2 g) l_1 \sin\theta_1$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 l_1^2 + m_2 l_2^2)\dot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_2$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 - m_2 g l_2 \sin\theta_2$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

The equations of motion are

θ_1 :
$$(m_1 l_1^2 + m_2 l_2^2)\ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) +$$
$$+ m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 + (m_1 g + m_2 g) l_1 \sin\theta_1$$

or

$$(m_1 l_1^2 + m_2 l_2^2)\ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 - m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2^2 + (m_1 + m_2) g l_1 \sin\theta_1 = 0$$

θ_2 :
$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + m_2 l_1 l_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1^2 - 2\dot{\theta}_1 \dot{\theta}_2) + m_2 g l_2 \sin\theta_2 = 0$$

(5) Goldstein 1.23

Our dissipation function is given by $D = \frac{1}{2} k v^2$

The Lagrangian is $L = T - V = \frac{1}{2} m \dot{y}^2 + mgy$ (we assume the y-axis points down)

The Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} + \frac{\partial D}{\partial \dot{y}} = 0$$

$$m\ddot{y} - mg + k\dot{y} = 0$$

or

$$m\dot{v} + kv - mg = 0 \qquad \text{where } v = \frac{dy}{dt}$$

If we take another derivative we obtain:

$$\ddot{v} + k\dot{v} = 0 \qquad \text{or} \qquad m\dot{a} + ka = 0 \qquad \text{where } a = \frac{dv}{dt}$$

$$\frac{da}{a} = -\frac{k}{m}dt \implies a(t) = Ae^{-\frac{k}{m}t} \qquad A = \text{const}$$

then we can integrate and get the velocity:

$$v(t) = C - \frac{Am}{k}e^{-\frac{k}{m}t} \qquad \text{where } C = \text{const}$$

Since $v(0) = 0$ $\quad C = \frac{Am}{k}$ and we get

$$v(t) = \frac{Am}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

Now at $t=0$ $\quad m\ddot{y} = mg$ $(k\dot{y}(0) = 0)$, which gives us $\dot{v}(0) = a(0) = g$, from where we can find $A = g$, so that $v(t) = \frac{gm}{k}\left(1 - e^{-\frac{k}{m}t}\right)$

at $t \to \infty$ $\quad v(t) \to \frac{gm}{k}$