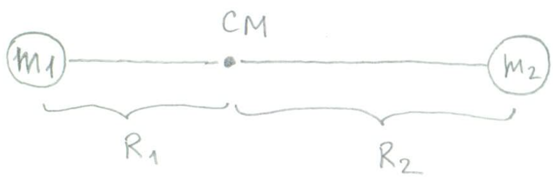
(2) Problem 3.11 in Goldstein



If we assume the center of mass of the system is at rest, each particle rotates around the CM at distance R_i and $R_1 + R_2 = R$. The magnitude of the force with which the particles act on each other is $\frac{\alpha}{R^2}$. Hence,

$$\frac{m_1 v_1^2}{R_1} = \frac{m_2 v_2^2}{R_2} = \frac{\alpha}{(R_1+R_2)^2}$$

where v_1 and v_2 are the rotational velocities of the particles. The period of the orbit is

$$\tau = \frac{2\pi R_1}{v_1}$$

while from the previous relation we have

$$v_1 = \frac{1}{R_1+R_2}\sqrt{\frac{\alpha R_1}{m_1}}$$

With that τ is

$$\tau = 2\pi R_1 (R_1+R_2)\sqrt{\frac{m_1}{R_1 \alpha}}$$

Now since CM is at rest $m_1 R_1 = m_2 R_2$ or $R_2 = \frac{m_1}{m_2} R_1$. Then

$$\tau = 2\pi\left(1+\frac{m_1}{m_2}\right) R_1 \sqrt{\frac{m_1 R_1}{\alpha}} = 2\pi \frac{m_1^{3/2}}{\mu} R_1^{3/2} \quad \text{where} \quad \mu = \frac{m_1 m_2}{m_1+m_2}$$

When the two particles are stopped and then released, at any subsequent moment of time the total mechanical energy is

$$E = -\frac{\alpha}{R} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{\alpha}{r_1+r_2} \quad \text{where } r_1 = r_1(t) \quad r_2 = r_2(t)$$

The total momentum is conserved, so $p_1 = -p_2$ and

$$-\frac{\alpha}{R} = \frac{1}{2\mu} p_1^2 - \frac{\alpha}{r_1+r_2}$$

Again, using the fact that the CM is at rest we have $r_2 = \frac{m_1}{m_2} r_1$

and the last equation can be rewritten as

$$\frac{\alpha \mu}{m_1}\frac{1}{r_1} - \frac{\alpha \mu}{m_1}\frac{1}{R_1} = \frac{1}{2\mu} p_1^2$$

Since $p_1 = m_1 v_1$ (v_1 here is the "radial" velocity of particle 1)

$$v_1 = \frac{dr_1}{dt} = \frac{\mu}{m_1}\sqrt{\frac{2\alpha}{m_1}}\sqrt{\frac{1}{r_1} - \frac{1}{R_1}}$$

or

$$dt = \frac{m_1^{3/2}}{\mu}\sqrt{\frac{1}{2\alpha}}\frac{dr_1}{\sqrt{\frac{1}{r_1} - \frac{1}{R_1}}}$$

The time it takes for particle 1 to "fall" from R_1 to 0 (and for particle 2 to "fall" from R_2 to 0) is

$$t = \frac{m_1^{3/2}}{\mu}\sqrt{\frac{1}{2\alpha}}\int_0^{R_1}\frac{dr_1}{\sqrt{\frac{1}{r_1} - \frac{1}{R_1}}} = \frac{m_1^{3/2}}{\mu}\sqrt{\frac{1}{2\alpha}}\ \frac{1}{2}\pi R_1^{3/2}$$

Comparing τ and t we can easily see that

$$\tau = 4\sqrt{2}\, t$$

(3) Here we can use formula (3.57) from Goldstein for the eccentricity:

$$\varepsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$$

assuming that the potential is $V(r) = -\frac{k}{r} = -G\frac{mM}{r}$ where m is the mass of the comet and M is the mass of the Sun.

The total energy is $E = \frac{mv^2}{2} - G\frac{mM}{r}$, while the angular momentum is given by:

$$\vec{l} = m(\vec{r} \times \vec{v}) \qquad l^2 = m^2 r^2 v^2 \sin^2\phi$$

Plugging E and l into the formula for eccentricity gives the final result

$$\varepsilon = \sqrt{1 + \left(v^2 - \frac{2GM}{r}\right)\left(\frac{r\, v \sin\phi}{GM}\right)^2}$$

Note that the mass of the comet cancels out

(4) Here we have $r(\theta) = \alpha\theta^2$ $\quad \frac{dr}{d\theta} = 2\alpha\theta$

From equation (3.33) of Goldstein, which reads

$$\frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} = f(r) \qquad \text{where } f(r) = -\frac{\partial V}{\partial r}$$

we obtain

$$\frac{l^2}{mr^2}\frac{d}{d\theta}\left(\frac{2\alpha\theta}{r^2}\right) - \frac{l^2}{mr^3} = f(r) \qquad \text{or} \qquad \frac{2\alpha l^2}{mr^2}\frac{d}{d\theta}\left(\frac{1}{\alpha^2\theta^3}\right) - \frac{l^2}{mr^3} = f(r)$$

then

$$-\frac{6l^2}{mr^2\alpha}\frac{1}{\theta^4} - \frac{l^2}{mr^3} = f(r)$$

$$-\frac{6l^2\alpha}{mr^4} - \frac{l^2}{mr^3} = f(r)$$

$$\frac{l^2}{m}\left(\frac{6\alpha}{r^4} + \frac{1}{r^3}\right) = \frac{\partial V}{\partial r}$$

$$V(r) = -\frac{l^2}{2m}\left(\frac{4\alpha}{r^3} + \frac{1}{r^2}\right) + \text{const}$$

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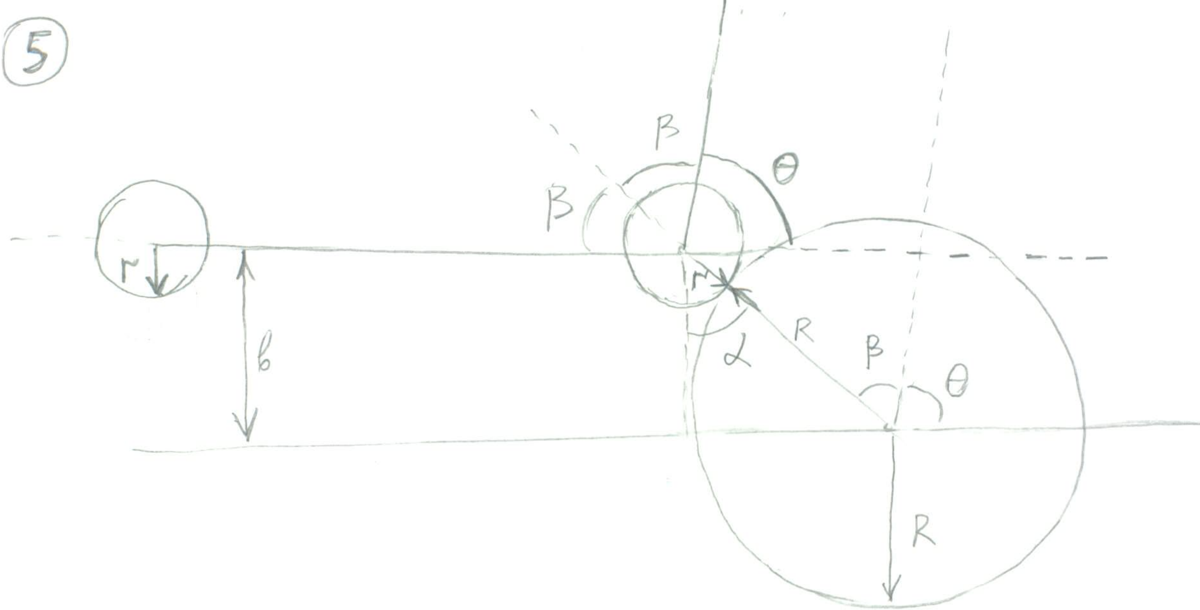

According to the definition of $\sigma(\theta)$ (see lecture notes):

$$2\pi b\, db = \pm \sigma(\theta) \cdot 2\pi \sin\theta\, d\theta$$

In our case the force acting on moving sphere m is repulsive, so we pick the minus sign ($\frac{db}{d\theta}$ is negative). We deal with a simple deflection of m off the surface of sphere M. From the figure above we can conclude that

$$(R+r)\cos\alpha = b$$

$$\frac{\pi}{2} - \alpha = \pi - \beta - \theta \quad \Rightarrow \quad \alpha = \beta + \theta - \frac{\pi}{2}$$

Also $2\beta + \theta = \pi \quad \Rightarrow \quad \beta = \frac{\pi - \theta}{2}$

Hence $\alpha = \frac{\theta}{2}$ and $b = (R+r)\cos\frac{\theta}{2}$ $\quad db = -\frac{R+r}{2}\sin\frac{\theta}{2}\, d\theta$

When we plug these expressions in the very first formula we obtain

$$-\pi (R+r)^2 \cos\frac{\theta}{2}\sin\frac{\theta}{2}\, d\theta = -\sigma(\theta)\cdot 2\pi \sin\theta\, d\theta$$

from which we can see that

$$\sigma(\theta) = \frac{\frac{1}{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2}}{\sin\theta}(R+r)^2 = \frac{1}{4}(R+r)^2 \quad \leftarrow$$

no dependence on θ !!!

The total cross section is

$$\sigma_{tot} = \int_{4\pi} \sigma(\theta)\, d\Omega = 4\pi \cdot \frac{1}{4}(R+r)^2 = \pi (R+r)^2$$

This result is expected since no scattering takes place for $b > R+r$