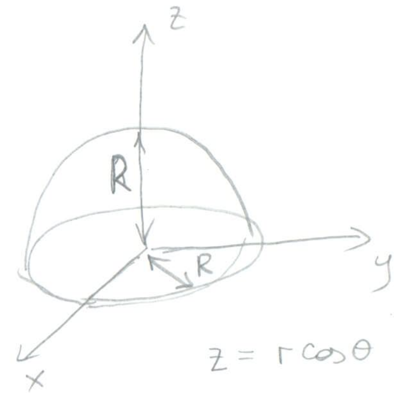
① Let us find the location of the center of mass. We place the origin in the center of the hemisphere's base. Due to the symmetry $y_{CM} = x_{CM} = 0$. For z_{CM} we get



$$z_{CM} = \frac{\int z\rho\, dV}{\int \rho\, dV} = \frac{\int_0^{2\pi} d\varphi \int_0^{\pi/2} \sin\theta\, d\theta \int_0^R r^2 dr\, r\cos\theta}{V} =$$

$$= \frac{1}{\frac{2}{3}\pi R^3} \cdot 2\pi \cdot \underbrace{\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta}_{1/2} \underbrace{\int_0^R r^3 dr}_{R^4/4} = \frac{3}{8}R$$

Let us also compute $I_{xx} = I_{yy} = I'_{zz}$ in the same coordinate frame (where the center of the base is at the origin). This is very easy to do if we use the symmetry: the moment of inertia of a sphere about its origin (regardless of the direction of the axis) is

$$I_{sphere} = \int \rho(x^2+y^2)dV = 2\int \rho z^2 dV = 2\rho\, 2\pi \cdot \underbrace{\int_0^{\pi} \sin\theta\cos^2\theta\, d\theta}_{2/3} \underbrace{\int_0^R r^4 dr}_{R^5/5} =$$

$$= \frac{8\pi\rho R^5}{15}$$

a hemisphere contributes exactly one half into it.

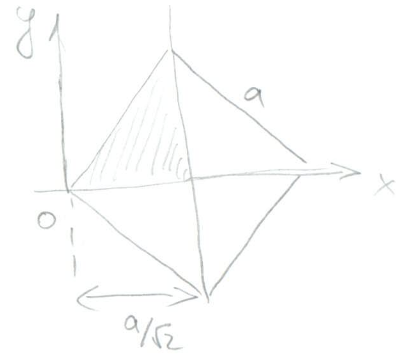
So
$$I_{xx} = I_{yy} = I_{zz} = \frac{4}{15}\pi\rho R^5$$

Now we can use the parallel axis theorem to find $I_1 = I_2$ in the new frame (where the origin is at the center of mass

$$I_1 = I_2 = I_{xx} - M z_{cm}^2 = \frac{4}{15}\pi\rho R^5 - \frac{2}{3}\pi R^3 \rho \cdot \left(\frac{3}{8}R\right)^2 = \frac{83}{480}\pi\rho R^5$$

I_3 is the same as I_{zz} because the shift by z_{cm} is along the z-axis: $I_3 = I_{zz} = \frac{4}{15}\pi\rho R^5$

(2) The kinetic energy of rotation is $\frac{1}{2} I \omega^2$ where I is the moment of inertia about the diagonal.



$$I = \underbrace{4}_{\text{we can split the plate into 4 equally contributing triangles}} \int y^2 \, dm$$

$dm = \sigma \, dx \, dy$ where $\sigma = \frac{m}{a^2}$

So

$$I = 4\sigma \int_0^{a/\sqrt{2}} dx \int_0^{x} y^2 \, dy = \frac{4m}{3a^2} \int_0^{a/\sqrt{2}} x^3 \, dx = \frac{m}{3a^2} \left(\frac{a}{\sqrt{2}}\right)^4 = \frac{ma^2}{12}$$

and

$$T = \frac{1}{2} I \omega^2 = \frac{m a^2 \omega^2}{24}$$

(3) See Goldstein chapter 4.7

④ a) The principal moments of inertia are (assuming that the z-axis passes through the center of the disk and is perpendicular to it)

$$I_3 = \int (x^2+y^2)\,dm \qquad \text{where } \sigma = \frac{m}{\pi R^2} \text{ - mass per}$$

$$= \int \sigma (x^2+y^2)\,dx\,dy$$

$$= \sigma \int_0^{2\pi} d\phi \int_0^R r^2\, r\,dr = \frac{m}{\pi R^2}\, 2\pi \frac{R^4}{4} = \frac{mR^2}{2}$$

Due to symmetry $I_1 = I_2$ and

$$I_1 = \int (y^2 + \underbrace{z^2}_{0})\,dm = \sigma \int_0^{2\pi} d\phi \int_0^R r^2 \sin^2\phi\, r\,dr = \frac{m}{\pi R^2} \cdot \pi \frac{R^4}{4} = \frac{mR^2}{4}$$

b) Euler's equations

$$\begin{cases} I_1 \dot\omega_1 + (I_3 - I_2)\omega_2\omega_3 = N_1 \\ I_2 \dot\omega_2 + (I_1 - I_3)\omega_1\omega_3 = N_2 \\ I_3 \dot\omega_3 + (I_2 - I_1)\omega_1\omega_2 = N_3 \end{cases}$$

Since the motion is said to be maintained (i.e. constrained about a fixed axis in the body frame) $\vec\omega = \text{const}$ $\quad \dot{\vec\omega} = 0$

Then the Euler equations become

$$\begin{cases} (I_3 - I_2)\omega_2\omega_3 = N_1 \\ (I_1 - I_3)\omega_1\omega_3 = N_2 \\ (I_2 - I_1)\omega_1\omega_2 = N_3 \end{cases} \quad \text{or} \quad \begin{cases} \frac{mR^2}{4}\omega_2\omega_3 = N_1 \\ -\frac{mR^2}{4}\omega_1\omega_3 = N_2 \\ 0 = N_3 \end{cases}$$

$$\omega_3 = \omega \cos\theta \qquad \sqrt{\omega_1^2 + \omega_2^2} = \omega \sin\theta$$

If we square and add the equations for N_1 and N_2 then

$$N_1^2 + N_2^2 = \frac{m^2R^4}{16}(\omega_1^2 + \omega_2^2)\omega_3^2 = \frac{m^2R^4}{16}\omega^4 \cos^2\theta \sin^2\theta$$

$$|\vec N| = \sqrt{N_1^2 + N_2^2} = \frac{m^2R^2\omega^2}{4} \cos\theta \sin\theta \qquad 0 < \theta < \frac{\pi}{2}$$

The individual values of N_1 and N_2 depend on the choice of x and y axes in the plane of the disk