(1) Following the same path as we used in lecture for an ideal forced harmonic oscillator, let us solve the equation

$$m\ddot{x} + b\dot{x} + kx = \delta(t) \qquad (1)$$

with the initial conditions $x(t) = 0, \; t < 0$

For $t > 0$ the solution of (1) $(\delta(t) = 0, \; t > 0)$ is

$$x(t) = A e^{-\frac{b}{2m}t} \sin\left(\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}\, t + \phi\right)$$ see lecture #20

Here we assumed weak damping $\left(\omega_0^2 > \frac{b^2}{4m^2}\right)$

Integration of

$$m\ddot{x} + b\dot{x} + kx = \delta(t)$$

from $-\epsilon$ to ϵ, where ϵ is an infinitely small number gives exactly the same condition as in the case of the ideal oscillator

$$m\int_{-\epsilon}^{\epsilon} \ddot{x}(t)\,dt + \underbrace{b\int_{-\epsilon}^{\epsilon} \dot{x}(t)\,dt}_{0} + \underbrace{k\int_{-\epsilon}^{\epsilon} x(t)\,dt}_{0} = \int_{-\epsilon}^{\epsilon} \delta(t)\,dt \Rightarrow m\dot{x}\Big|_{-\epsilon}^{\epsilon} = 1$$

$$m\dot{x}(t=0^+) - \underbrace{m\dot{x}(t=0^-)}_{0} = 1 \quad \Rightarrow \quad \dot{x}(t=0^+) = \frac{1}{m}$$

So we have $x(t=0^+) = 0$ and $\dot{x}(t=0^+) = \frac{1}{m}$ and that gives us $\phi = 0$ and

$$A\left(\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}\right) = \frac{1}{m} \quad \Rightarrow \quad A = \frac{1}{m\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}}$$

Therefore

$$x(t) = \begin{cases} \dfrac{e^{-\frac{b}{2m}t} \sin\left(\sqrt{\omega_0^2 - \frac{b^2}{4m}}\, t\right)}{m\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}}, & t > 0 \\ 0, & t < 0 \end{cases}$$

More generally for $F(t) = \delta(t-t')$ we will have

$$x(t) = \begin{cases} \dfrac{e^{-\frac{\beta}{2}(t-t')} \sin[\omega'(t-t')]}{m\omega'} , & t>0 \\ 0 , & t<0 \end{cases}$$

where $\beta = \frac{b}{m}$ and $\omega' = \sqrt{\omega_0^2 - \frac{\beta^2}{4}}$

Similarly to the lecture notes, we find that the particular solution for an arbitrary $F(t)$ is

$$x_p(t) = \int_{-\infty}^{t} F(t') \frac{e^{-\frac{\beta}{2}(t-t')} \sin(\omega'(t-t'))}{m\omega'} dt'$$

If $F(t) = \begin{cases} \alpha t , & t \geq 0 \\ 0 , & t<0 \end{cases}$ then $x_p(t)$ becomes

$$x_p(t) = \alpha \int_{-\infty}^{t} t' \frac{e^{-\frac{\beta}{2}(t-t')} \sin(\omega'(t-t'))}{m\omega'} dt' =$$

$$= \frac{\alpha}{m} \frac{t(\omega'^2 + \frac{\beta^2}{4}) - \beta}{(\omega'^2 + \frac{\beta^2}{4})^2}$$

Obviously $x_p(t)$ will dominate over the general (oscillatory and damped) solution of

$$m\ddot{x} + b\dot{x} + kx = 0$$

Therefore in the limit $t \to 0$

$$x(t) \to \frac{\alpha}{m} \frac{t}{\omega'^2 + \frac{\beta^2}{4}} = \frac{\alpha t}{m\left(\omega_0^2 - \frac{b^2}{4m^2} + \frac{b^2}{4m^2}\right)} = \frac{\alpha t}{k}$$

② $V(r) = -\frac{\alpha}{r} + \frac{\beta}{r^9}$

The minimum of $V(r_e)$ occurs when $\left.\frac{\partial V}{\partial r}\right|_{r_e} = 0$, so

$$\frac{\alpha}{r_e^2} - \frac{9\beta}{r_e^{10}} = 0 \quad \Rightarrow \quad r_e^8 = \left(\frac{9\beta}{\alpha}\right) \qquad r_e = \left(\frac{9\beta}{\alpha}\right)^{1/8}$$

β in terms of r_e is given by $\beta = \frac{\alpha r_e^8}{9}$

To find the frequency of small oscillations we need to Taylor expand $V(r)$ at $r = r_e$ up to the second order:

$$V(r) = V(r_e) + \underbrace{V'(r_e)}_{0}(r - r_e) + \frac{1}{2}\underbrace{V''(r_e)}_{k \leftarrow \text{"spring" constant}}(r - r_e)^2 + \ldots$$

$$V''(r) = -\frac{2\alpha}{r_e^3} + \frac{90\beta}{r_e^{11}} = -\frac{2\alpha}{r_e^3} + \frac{10\alpha}{r_e^3} = \frac{8\alpha}{r_e^3} = \frac{8\alpha^{11/8}}{(9\beta)^{3/8}} = k$$

So

$$\omega_0 = \sqrt{\frac{k}{\mu}} = \left[\frac{8\alpha^{11/8}}{\mu(9\beta)^{3/8}}\right]^{1/2}$$

3

The Lagrangian of this system is obviously

$$L = T - V = \frac{m}{2}\dot{x}_1^2 + \frac{M}{2}\dot{x}_2^2 + \frac{m}{2}\dot{x}_3^2 - \frac{k}{2}(x_2-x_1)^2 - \frac{k}{2}(x_3-x_2)^2$$

The equations of motions that follow from it are

$$m\ddot{x}_1 + kx_1 - kx_2 = 0$$

$$M\ddot{x}_2 - kx_1 + 2kx_2 - kx_3 = 0 \qquad \text{or} \qquad \mathbb{M}\ddot{\vec{x}} = -\mathbb{K}\vec{x}$$

$$m\ddot{x}_3 - kx_2 + kx_3 = 0$$

where

$$\mathbb{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \qquad \mathbb{K} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

We use the solution ansatz in the form

$$\vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{i\omega t}$$

which results in the eigenvalue problem $\mathbb{K}\vec{a} = \omega^2 \mathbb{M}\vec{a}$

The roots $\omega^2_{1,2,3}$ are found from the secular equation

$$\det(\mathbb{K} - \omega^2 \mathbb{M}) = \begin{vmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{vmatrix} = 0$$

or

$$\omega^2(-m\omega^2 + k)(-mM\omega^2 + km + 2km) = 0$$

The solutions are

$$\omega_1^2 = 0 \qquad \omega_2^2 = \frac{k}{m} \qquad \omega_3^2 = \frac{k}{m} + 2\frac{k}{M}$$

The first mode corresponds to no oscillation. It is purely translattenal motion and we can easily guess (of find) the eigenvector : $\vec{a}^{(1)} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$

The second mode correspond to $\omega_2 = \sqrt{\frac{k}{m}}$. It is not difficult to guess it corresponds to the situation where M is stationary while m's vibrate in the opposite direction, so $\vec{a}^{(2)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

Finally the third eigenvector corresponding to $\omega_3^2 = \frac{k}{m} + 2\frac{k}{M}$ is

$$\begin{pmatrix} -2\frac{m}{M}k & -k & 0 \\ -k & -\frac{M}{m}k & -k \\ 0 & -k & -2\frac{m}{M}k \end{pmatrix} \begin{pmatrix} a_1^{(3)} \\ a_2^{(3)} \\ a_3^{(3)} \end{pmatrix} = 0$$

$$2\frac{m}{M} a_1^{(3)} = a_2^{(3)}$$

$$a_1^{(3)} = a_3^{(3)}$$

so $$\vec{a} = \alpha \begin{pmatrix} 1 \\ -2\frac{m}{M} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}\sqrt{1+2\frac{m^2}{M^2}}} \begin{pmatrix} 1 \\ -2\frac{m}{M} \\ 1 \end{pmatrix}$$

This mode corresponds to a symmetric motion of m's while M moves in opposite direction with a different amplitude.

(4) First bob:

$x_1 = l\sin\theta_1$ $\quad \dot{x}_1 = l\cos\theta_1\dot{\theta}_1$ $\quad T_1 = \frac{m}{2}(\dot{x}_1^2+\dot{y}_1^2) = \frac{m}{2}l^2\dot{\theta}_1^2$

$y_1 = l\cos\theta_1$ $\quad \dot{y}_1 = -l\sin\theta_1\dot{\theta}_1$ $\quad V_1 = mgy_1 = mgl\cos\theta_1$

Second bob:

$x_2 = x_1 + l\sin\theta_2$ $\quad \dot{x}_2 = l(\cos\theta_1\dot{\theta}_1 + \cos\theta_2\dot{\theta}_2)$

$y_2 = y_1 + l\cos\theta_2$ $\quad \dot{y}_2 = -l(\sin\theta_1\dot{\theta}_1 + \sin\theta_2\dot{\theta}_2)$

$$T_2 = \frac{m}{2}[\dot{x}_2^2+\dot{y}_2^2] = \frac{ml^2}{2}\left[\dot{\theta}_1^2+\dot{\theta}_2^2+2(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)\dot{\theta}_1\dot{\theta}_2\right] =$$

$$= \frac{ml^2}{2}\left[\dot{\theta}_1^2+\dot{\theta}_2^2+2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\right]$$

$$V_2 = mgy_2 = mgl(\cos\theta_1+\cos\theta_2)$$

The Lagrangian of the whole system is:

$$L = \frac{ml^2}{2}\left[2\dot{\theta}_1^2+\dot{\theta}_2^2+2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\right] + mgl\left[2\cos\theta_1+\cos\theta_2\right]$$

If we assume both θ_1 and θ_2 to be small, Taylor expand L, and keep only the leading (non-constant) terms in the kinetic and potential energy, then the Lagrangian becomes

$$L = \frac{ml^2}{2}\left[2\dot{\theta}_1^2+\dot{\theta}_2^2+2\dot{\theta}_1\dot{\theta}_2\right] - mgl\left[\theta_1^2+\frac{\theta_2^2}{2}\right]$$

The equations of motion:

θ_1: $\quad 2ml^2\ddot{\theta}_1 + ml^2\ddot{\theta}_2 = -2mgl\theta_1$

θ_2: $\quad ml^2\ddot{\theta}_2 + ml^2\ddot{\theta}_1 = -mgl\theta_2$

If we denote $\frac{g}{l} = \omega_0^2$ we can write them in the matrix form

$$\underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}_{M}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\underbrace{\begin{pmatrix} 2\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{pmatrix}}_{\mathbb{K}}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

We will now seek the solution in the form

$$\vec{\theta} = \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$$

Then

$$\begin{pmatrix} 2\omega_0 & 0 \\ 0 & \omega_0^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \omega^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\det\left(K - \omega^2 M\right) = \begin{vmatrix} 2(\omega_0^2 - \omega^2) & -\omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 0$$

This yields an algebraic equation

$$2(\omega_0^2 - \omega^2)^2 - \omega^4 = \omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0$$

Its solutions are $\omega_{1,2}^2 = (2 \pm \sqrt{2})\omega_0^2$

The eigenvector corresponding to $\omega_1^2 = (2+\sqrt{2})\omega_0^2$ is

$$\begin{pmatrix} -2-2\sqrt{2} & -2-\sqrt{2} \\ -2-\sqrt{2} & -1-\sqrt{2} \end{pmatrix} \begin{pmatrix} a_1^{(1)} \\ a_2^{(1)} \end{pmatrix} = 0 \Rightarrow (2+\sqrt{2})a_1^{(1)} = (1+\sqrt{2})a_2^{(2)}$$

$$\vec{a}^{(1)} = \alpha \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix}$$

The eigenvector corresponding to $\omega_2^2 = (2-\sqrt{2})\omega_0^2$

$$\begin{pmatrix} -2-2\sqrt{2} & -2+\sqrt{2} \\ -2+\sqrt{2} & -1+\sqrt{2} \end{pmatrix} \begin{pmatrix} a_1^{(2)} \\ a_2^{(2)} \end{pmatrix} \qquad (-2+\sqrt{2})a_1^{(2)} = (1-\sqrt{2})a_2^{(2)}$$

$$\vec{a}^{(2)} = \beta \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$$

The general solution is a sum of two normal modes:

$$\vec{\theta} = \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = A \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix} e^{i(2+\sqrt{2})\omega_0 t} + B \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} e^{i(2-\sqrt{2})\omega_0 t}$$