(1) a) Since the force is spherically symmetric the motion will take place in a plane. In the polar coordinates the kinetic energy is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

The potential energy is (up to an arbitrary constant)

$$V = -\int F\,dr = -\frac{\alpha}{r}e^{-\beta t}$$

so the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\alpha}{r}e^{-\beta t}$$

Then

$$p_r \equiv \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}^2 \implies \dot{r} = \frac{p_r}{m} \qquad \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$H = \sum_i p_i\dot{q}_i - L = p_r\frac{p_r}{m} + p_\theta\frac{p_\theta}{mr^2} - \frac{1}{2}m\left(\frac{p_r^2}{m^2} + r^2\frac{p_\theta^2}{m^2r^4}\right) - \frac{\alpha}{r}e^{-\beta t} =$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{\alpha}{r}e^{-\beta t}$$

b) Hamilton's equations of motion

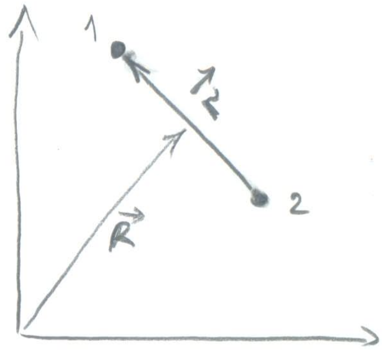
$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{\alpha}{r^2}e^{-\beta t} + \frac{p_\theta^2}{mr^3} \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

c) The energy is obviously not conserved because H has explicit time dependence.

(2) a) In this system it is convenient to use the center of mass coordinates and relative position:

$$\begin{cases} \vec{R} = \dfrac{\vec{r}_1 + \vec{r}_2}{2} \\ \vec{r} = \vec{r}_1 - \vec{r}_2 \end{cases} \qquad \begin{aligned} \vec{r}_1 &= \vec{R} + \frac{\vec{r}}{2} \\ \vec{r}_2 &= \vec{R} - \frac{\vec{r}}{2} \end{aligned}$$



The kinetic energy is

$$T = \frac{m}{2}\dot{\vec{r}}_1^{\,2} + \frac{m}{2}\dot{\vec{r}}_2^{\,2} = \frac{m}{2}\left(\dot{\vec{R}} + \frac{\dot{\vec{r}}}{2}\right)^2 + \frac{m}{2}\left(\dot{\vec{R}} - \frac{\dot{\vec{r}}}{2}\right)^2 =$$

$$= m\dot{\vec{R}}^2 + \frac{m}{4}\dot{\vec{r}}^{\,2}$$

For $\vec{R}$ we can use the Cartesian components $\vec{R} = (X, Y)$, while for $\vec{r}$ it is more natural to choose the polar coordinates r and θ. Then

$$T = m(\dot{X}^2 + \dot{Y}^2) + \frac{m}{4}(\dot{r} + r^2\dot{\theta}^2)$$

The potential energy obviosly depends on r only:

$$V = \frac{1}{2}k(r - \ell)^2$$

Hence our Lagrangian is

$$L = m(\dot{X}^2 + \dot{Y}^2) + \frac{m}{4}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}k(r-\ell)^2$$

To determine the Hamiltonian we follow the standard procedure.

$$P_X = \frac{\partial L}{\partial \dot{X}} = 2m\dot{X} \quad \Rightarrow \quad \dot{X} = \frac{P_X}{2m}$$

$$P_Y = \frac{\partial L}{\partial \dot{Y}} = 2m\dot{Y} \quad \Rightarrow \quad \dot{Y} = \frac{P_Y}{2m}$$

$$P_r = \frac{\partial L}{\partial \dot{r}} = \frac{m}{2}\dot{r} \quad \Rightarrow \quad \dot{r} = \frac{2p_r}{m}$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{m}{2}r^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{2p_\theta}{mr^2}$$

$$H = P_X\dot{X} + P_Y\dot{Y} + p_r\dot{r} + p_\theta\dot{\theta} - L = \frac{P_X^2}{4m} + \frac{P_Y^2}{4m} + \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{1}{2}k(r-\ell)^2$$

b) Hamilton's equations of motion

$$\dot{X} = \frac{\partial H}{\partial P_X} = \frac{P_x}{2m} \qquad \dot{P}_x = -\frac{\partial H}{\partial X} = 0$$

$$\dot{Y} = \frac{\partial H}{\partial P_Y} = \frac{P_Y}{2m} \qquad \dot{P}_y = -\frac{\partial H}{\partial Y} = 0$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{2p_r}{m} \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{2p_\theta^2}{mr^3} + k(r-\ell)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{2p_\theta}{mr^2} \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

c) From the right column in part b) we can see that P_x, P_Y, and p_θ are conserved. They correspond to cyclic coordinates X, Y, and θ

(3) For a canonical transformation the Poisson bracket $\{P, Q\}$ must be equal to 1:

$$\{P,Q\} = \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = 0 \cdot \left(-\frac{\alpha p}{q^2}\right) - 2\beta q \cdot \frac{\alpha}{q} = -2\beta\alpha = 1$$

Therefore the condition is

$$\alpha\beta = -\frac{1}{2}$$

(4) Similarly to the previous problem, we require

$$\{P, Q\} = 1$$

In this case it gives

$$\{P,Q\} = \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = \beta q^{\alpha} \cos(\beta p)\, \alpha q^{\alpha-1} \cos(\beta p) +$$

$$+ \alpha q^{\alpha-1} \sin(\beta p)\, q^{\alpha} \beta \sin(\beta p) = \alpha\beta q^{2\alpha-1} = 1$$

We have to require $\alpha = \frac{1}{2}$ and $\beta = 2$