Brief review of Newtonian mechanics

$\vec{p} = m\vec{V}$ linear momentum

$\vec{V} = \dfrac{d\vec{r}}{dt}$ velocity

$\vec{a} = \dfrac{d\vec{V}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$ acceleration

First Newton's law: if $\sum \vec{F} = 0$ then $\dfrac{d\vec{V}}{dt} = 0$

$\vec{V} = \text{const}$

Second Newton's law: $\vec{F} = m\vec{a}$ or $\vec{F} = \dfrac{d\vec{p}}{dt} = \dfrac{d(m\vec{V})}{dt}$

Inertial frame of reference (Galilean frame) is where time and space is homogeneos and isotropic, and where $\vec{F} = \dfrac{d\vec{p}}{dt}$ holds true.

During the motion of a mechanical system, the quantities that specify its state (generalized coordinates and generalized velocities) may vary with time. However, there may be functions of those quantities which remain constant during the motion. Such functions are called integrals of motion

The number of independent integrals of motion for a closed system with n degrees of freedom is $2n-1$. This follows from the fact that the general solution of the equations of motion contains $2n$ arbitrary constants. Since the equations of motion for a closed system do not involve the time explicitly the choice of the origin of time is entirely arbitrary. One of the arbitrary constants in the solution of the equations can always be taken as an additive constant t_0 in the time. Then eliminating $t+t_0$ from $2n$ functions $x_i = x_i(t+t_0, C_1, \dots C_{2n-1})$, $\dot{x}_i = \dot{x}_i(t+t_0, C_1, \dots C_{2n-1})$ we can express the $2n-1$ arbitrary constants as functions of x and $\dot{x}$ and these functions will be integrals of motion.

Linear momentum:

if the total force $\vec{F}$ is zero then $\dot{\vec{p}} = 0 \Rightarrow \vec{p} = \text{const}$

Angular momentum: $\vec{L} = \vec{r} \times \vec{p}$

define torque $\vec{N} = \vec{r} \times \vec{F}$ then $\vec{r} \times \vec{F} = \vec{r} \times \frac{d}{dt}(m\vec{v})$

and using the identity $\frac{d}{dt}(\vec{r} \times m\vec{v}) = \underbrace{\vec{v} \times m\vec{v}}_{0} + \vec{r} \times \frac{d}{dt}(m\vec{v})$

we can conclude that

$$\vec{N} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = \frac{d\vec{L}}{dt} = \dot{\vec{L}}$$

Hence, whe the total torque, $\vec{N}$, is zero then $\vec{L} = \text{const}$.

Work: define $W_{12} = \int_1^2 \vec{F} \cdot d\vec{r}$

if $m = \text{const}$

$$\int \vec{F} \cdot d\vec{r} = m \int \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt = \frac{m}{2} \int \frac{d}{dt}(v^2)\, dt \Rightarrow W_{12} = \frac{m}{2}(v_2^2 - v_1^2)$$

$\frac{mv^2}{2}$ – kinetic energy of the particle

$$W_{12} = T_2 - T_1$$

If W_{12} is the same regardless of the path chosen betwen points 1 and 2 the force $(\vec{F})$ is called conservative. In other words, for a conservative force $\oint \vec{F} \cdot d\vec{r} = 0$

It can be shown that if $\vec{F}$ can be represented as a gradient of a scalar function $V(\vec{r})$ then $\vec{F}$ is conservative: $\vec{F} = -\nabla V(\vec{r})$

$$\oint \vec{F}\cdot d\vec{r} = -\oint \nabla V \cdot d\vec{r} \;\underset{\text{Kelvin-Stokes theorem}}{=\!=\!=}\; \int_A \nabla\times(\nabla V)\cdot d\vec{a} = 0$$

A - surface enclosed by loop

$$\nabla\times(\nabla f) \equiv 0 \quad \text{for any function } f$$

V is called the potential energy

For a conservative system $W_{12} = V_1 - V_2$

and $T_1 + V_1 = T_2 + V_2$

Energy conservation : if forces acting on a particle are conservative its energy is conserved

Systems of many particles

Second Newton law :

$$\underbrace{\sum_j{}' \vec{F}_{ji}}_{\text{force due to interaction of particle } i \text{ an } j} + \underbrace{\vec{F}_i^{ext}}_{\text{external force}} = \dot{p}_i$$

If we assume $\vec{F}_{ij} = -\vec{F}_{ji}$ (weak law of action and reaction)

then the above equation becomes when we sum over i

$$\sum_{j,i} F_{ji} + \sum_i F_i^{ext} = \frac{d^2}{dt^2}\sum_i m_i r_i$$

Now we can define the center of mass as

$$\vec{R} = \frac{\sum_i' m_i \vec{r}_i}{\sum_i m_i} = \frac{1}{M}\sum_i m_i \vec{r}_i$$

Using the definition of the center of mass, $\vec{R}$ we can write

$$M \frac{d^2 \vec{R}}{dt^2} = \sum_i \vec{F}_i^{ext} \equiv F^{ext}$$

which looks identical as the equation of motion of a single particle.

The concept of the center of mass can be easily generalized to continuous mass distribution. If we, instead of dealing with a bunch of point masses, assume some density $\rho(\vec{r})$ then

$$\vec{R} = \frac{\int \rho(\vec{r}) \vec{r} \, d\vec{r}}{\int \rho(\vec{r}) \, d\vec{r}} \qquad \int \rho(\vec{r}) \, d\vec{r} = M$$

The total linear momentum of a system is defined as

$$\vec{P} = \sum m_i \frac{d\vec{r}_i}{dt} = M \frac{d\vec{R}}{dt} \qquad \left(\vec{P} = \int \rho(\vec{r}) \vec{v}(\vec{r}) \, d\vec{r} \text{ for continuous mass distr} \right)$$

Again, the equation looks the same as the one for a single particle. It follows from here that if the total external force is zero, the total linear momentum is conserved

Let us recall the definition of the angular momentum of a particle:

$$\vec{l}_i = r_i \times \vec{p}_i$$

The total angular momentum of the system is then

$$\sum_i \vec{r}_i \times \dot{\vec{p}}_i = \sum_i \frac{d}{dt} (\vec{r}_i \times \vec{p}_i) = \dot{\vec{L}} = \sum_i \vec{r}_i \times F_i^{ext} + \sum_{i,j} r_i \times \vec{F}_{ji}$$

The last term can be considered a sum of the pairs

$$\vec{r}_i \times \vec{F}_{ji} + r_j \times \vec{F}_{ij} = (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji}$$

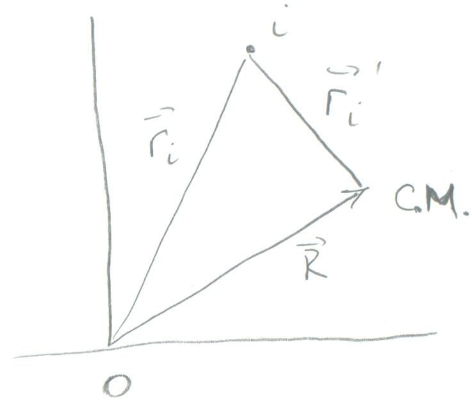
when we assume the equality of action and reaction.
If the internal forces between particles j and i, in addition to being equal and opposite also lie along the line joining the particles (strong law of action and reaction) then all $\vec{r}_{ij} \times \vec{F}_{ji}$ vanish.

Then

$$\frac{d\vec{L}}{dt} = \vec{N}^{\,external}$$ — The total angular momentum is constant if the external torque is zero.

Now let us pick some origin as a reference point. The total angular momentum of the system with respect to O is

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$$



let $\vec{r}_i'$ be the radius-vector from the center of mass to the i-th particle.

$$\vec{r}_i = \vec{r}_i' + \vec{R} \qquad \vec{v}_i = \vec{v}_i' + \underbrace{\vec{V}}_{\frac{d\vec{R}}{dt}}$$

$\frac{d\vec{R}}{dt}$ – velocity of the C.M relative to O

$\vec{v}_i'$ – velocity of the i-th particle relative to C.M.

Then

$$\vec{L} = \sum_i \vec{R} \times m_i \vec{V} + \sum_i \vec{r}_i' \times m_i \vec{v}_i' + \underbrace{\left(\sum_i m_i \vec{r}_i'\right)}_{0} \times \vec{V} + \vec{R} \times \frac{d}{dt} \underbrace{\sum_i m_i \vec{r}_i'}_{0}$$

or

$$\vec{L} = \underbrace{\vec{R} \times M\vec{v}}_{\text{angular momentum of the center of mass}} + \underbrace{\sum_i{}' \vec{r}_i \times \vec{p}_i{}'}_{\text{angular momentum of the motion about the center of mass}}$$

Lastly let us consider the work and energy. The work done by all forces is moving the system from mechanical state 1 to mechanical state 2

$$W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{r}_i = \sum_i \int_1^2 \vec{F}_i^{ext} \cdot d\vec{r}_i + \sum_{i,j}{}' \int_1^2 \vec{F}_{ji} \cdot d\vec{r}_i$$

this can be reduced to

$$\sum_i \int_1^2 \vec{F}_i \cdot d\vec{r}_i = \sum_i \int_1^2 m_i \dot{\vec{v}}_i \cdot \vec{v}_i \, dt = \sum_i{}' \int_1^2 d\left(\tfrac{1}{2} m_i v_i^2\right)$$

Hence

$$W_{12} = T_2 - T_1 \qquad \text{where} \quad T = \frac{1}{2} \sum_i{}' m_i v_i^2$$

since $\vec{v}_i = \vec{v}_i{}' + \vec{v}$ we have

$$T = \frac{1}{2} \sum_i m_i (\vec{v} + \vec{v}_i{}') \cdot (\vec{v} + \vec{v}_i{}') = \frac{1}{2} \sum_i{}' m_i v^2 + \frac{1}{2} \sum_i{}' m_i v_i{}'^2 + \vec{v} \cdot \frac{d}{dt} \underbrace{\left(\sum_i m_i \vec{r}_i{}'\right)}_{0}$$

The kinetic energy, like the angular momentum, also contains two terms: the kinetic energy of the center of mass and the kinetic energy of the motion about the center of mass

When the external forces are derivable from a potential

$$\sum_i{}' \int_1^2 F_i^{ext} \cdot d\vec{r}_i = -\sum_i{}' \int_1^2 (\nabla_i V_i) \cdot d\vec{r}_i = -\sum_i{}' V_i \Big|_1^2$$

If the internal forces are also conservative then

F_{ij} and F_{ji} can be obtained from a potential function V_{ij}. To satisfy the strong law of action and reaction V_{ij} must be central:

$$V_{ij} = V_{ij}(|\vec{r}_i - \vec{r}_j|)$$

$$\vec{F}_{ji} = -\nabla_i V_{ij} = \nabla_j V_{ij} = -\vec{F}_{ij}$$

$$\nabla V_{ij}(|\vec{r}_i - \vec{r}_j|) = (\vec{r}_i - \vec{r}_j) f(|\vec{r}_i - \vec{r}_j|)$$

When the forces are all conservative

$$\sum_{i \neq j} \int_1^2 \vec{F}_{ji} \cdot d\vec{r}_i = -\sum_{i,j} (\nabla_i V_{ij} \cdot d\vec{r}_i + \nabla_j V_{ij} \cdot d\vec{r}_j)$$

Introducing the notations

$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j \qquad \nabla_{ij} \equiv \nabla_{\vec{r}_{ij}} \qquad \nabla_i V_{ij} = \nabla_{ij} V_{ij} = -\nabla_j V_{ij}$$

$$d\vec{r}_i - d\vec{r}_j \equiv d\vec{r}_{ij}$$

we get

$$-\int \nabla_{ij} V_{ij} \cdot d\vec{r}_{ij}$$

The total work then reduces to

$$-\frac{1}{2} \sum_{i \neq j}{}' \int_1^2 \nabla_{ij} V_{ij} \cdot d\vec{r}_{ij} = -\frac{1}{2} \sum_{i,j}{}' V_{ij} \Big|_1^2$$

factor 1/2 is due to each given pair appearing twice

It turns out that if both the external and internal forces are derivable from potentials it is possible to define the total potential energy of the system

$$V = \sum_i V_i + \frac{1}{2} \sum_{i,j} V_{ij}$$

such that the total energy is conserved