D'Alembert's principle and Lagrange's equations

Many, if not most, mechanical systems contain constrains that limit the motion. Constraints may be classified in various ways. We will use the following classification:

1. Holonomic constraints are those that can be expressed in the form of equation(s)

$$g_i(\vec{r}_1, \ldots, \vec{r}_n, t) = 0 \qquad [\text{strict } =]$$

2. Non-holonomic constraints are such that are not expressible in the above form. They may sometimes be cast in the form of inequalities

$$g_i(\vec{r}_1, \ldots \vec{r}_n, t) \geq 0$$

Constraints are further classified according to whether they involve the time explicitly (rheonomous) or do not involve the time explicitly (scleronomous).

Constraints cause two types of complications when solving mechanical problems. First, coordinates $\vec{r}_i$ are no longer all independent. Second, they impose certain forces that cannot be specified directly.

To deal with holonomic constraints it is convenient to use generalized coordinates. If we have k constraints and $3N$ Cartesian coordinates then we may use the equations of constraints to eliminate k coordinates:

$$q_1 = q_1(\vec{r}_1, \ldots \vec{r}_N, t)$$

$$\vdots$$

$$q_{3N-k} = q_{3N-k}(\vec{r}_1, \ldots \vec{r}_N, t)$$

and

$$\vec{r}_1 = \vec{r}_1(q_1, \dots, q_{3N-k}, t)$$

$$\vdots$$

$$\vec{r}_N = \vec{r}_N(q_1, \dots, q_{3N-k}, t)$$

These transformation equations from $\vec{r}_i$ to q_i can be considered as parametric representations of $\vec{r}_i$ variables.

Generalized coordinates are often convenient even in the systems with no constraints due to the symmetry of the interaction. Example is a particle moving in a spherically symmetric force field.

If the constraint is non-holonomic the equations expressing the constraints cannot be used to eliminate the dependent variables. They can be applied only after the problem is solved.

The holonomic constraints are easier to deal with. The dependent equations can always be eliminated by the method of Lagrange multipliers. Nonholonomic constraints do not allow a general appoach and must be tackled individually

To deal with the second type of difficulty, that the forces of constraint are unknown a priori it is possible to introduce a kind of formalism where forces of constraint disappear.

Let us introduce the idea of a virtual (infinitesimal) displacement, $\delta\vec{r}_i$. This displacement is consistent with the forces and constraints at the given istant t. The displacement is called virtual to distinguish it from an actual displacement of the system occuring in a time interval dt, during which the forces and constraints may be changing.

Suppose the system is in equilibrium and the total force on each particle vanishes, $\vec{F}_i = 0$. Then $\vec{F}_i \cdot \delta\vec{r}_i = 0$ and the sum over all particles is also zero: $\sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0$. If we decompose $\vec{F}_i$ into the applied force $\vec{F}_i^{(a)}$ and the force of constraint $\vec{f}_i$, i.e. $\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i$ then

$$\sum_i F_i^{(a)} \cdot \delta\vec{r}_i + \sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0$$

We will now restrict ourselves to the case when the net virtual work of the forces of constraint is zero (examples: rigid body, a particle on a horizontal surface where $\vec{f} \cdot \delta\vec{r} = 0$ becase $\vec{f} \perp \delta\vec{r}$). We therefore have as the condition for equilibrium of a system

$$\sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i = 0$$

here $\delta\vec{r}_i$ are connected by constraints and $\vec{F}_i^{(a)} \neq 0$. n order to equate the coefficients in front of the orces to zero we must transform into virtual displacements of q_i, which are independent.

Recall the equation of motion $\vec{F}_i = \dot{p}_i$. Rather an $\sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0$ we can write

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0$$

..., by making the same resolution into applied forces and forces of constraint

$$\sum_i{}' (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i + \sum_i \vec{f}_i \cdot \delta \vec{r}_i = 0$$

Again we will assume that the virtual work of the forces of constrain vanishes, i.e. $\sum_i \vec{f}_i \cdot \delta \vec{r}_i = 0$. Then

$$\sum{}' (F_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0 \quad \leftarrow \text{D'Alembert principle}$$

Now let us transform from $\vec{r}_i$ to q_i:

$$\vec{r}_i = \vec{r}_i(q_1, \ldots q_n, t) \quad \text{— assume } n \text{ independent coordinates}$$

The velocities are

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \sum_k{}' \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}$$

An arbitrary virtual displacement is given by

$$\delta \vec{r}_i = \sum_j{}' \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

$\leftarrow$ no δt is here since a virtual displacement considers the displacement of coordinates

Then the virtual work of $F_i^{(a)}$ becomes (from now on we will drop the (a) superscript in $F_i^{(a)}$)

$$\sum_i \vec{F}_i \cdot \delta \vec{r}_i = \sum_{i,j} F_i \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j \equiv \sum_j Q_j \delta q_j$$

Here $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$ is generalized force

On the other hand

$$\sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_i{}' m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i = \sum_{i,j}{}' m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j \qquad (*)$$

Now

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i{}' \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right] \qquad (**)$$

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_j} = \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t} = \frac{\partial \vec{v}_i}{\partial q_j}$$

From

$$\vec{v}_i = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}$$ it also follows that

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$$

Substituting the above relations in (**) leads to

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_i}\right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j}\right]$$

and the term $\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i$ in $\sum_i (F_i^{(a)} - \dot{\vec{p}}_i)\cdot\delta\vec{r}_i = 0$ (see (*))

can be expressed as

$$\sum_j \left\{\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right)\right] - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right) - Q_j\right\}\delta q_j$$

Then if we use $T = \frac{1}{2}\sum_i m v_i^2$ D'Alembert's principle becomes

$$\sum_j \left\{\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] - Q_j\right\}\delta q_j = 0$$

The term $\frac{\partial T}{\partial q_j}$ is zero in Cartesian coordinates. However, in curvilinear ones it is generally not zero

If we now assume that the constraints we deal with are holonomic (in addition to $\sum_i f_i \cdot \delta\vec{r}_i = 0$) then q_j can be chosen in such a way that they contain constraints implicitly. Any virtual displacement δq_j is then independent of δq_k $(k \neq i)$ an we may state tha

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \qquad j = 1, \ldots, n$$

When the forces can be represented as a gradient of V, i.e. $\vec{F}_i = -\nabla_i V$ then the generalized force can be written as

$$Q_j = \sum_i{}' \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i{}' \nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} \equiv -\frac{\partial V}{\partial q_j}$$

Then our equation for T can be rewritten as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

The way we defined V it does not depend on generalized velocities. Thus, we can include it to the first term of our equation

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

Defining the Lagrangian as $L = T - V$ it becomes

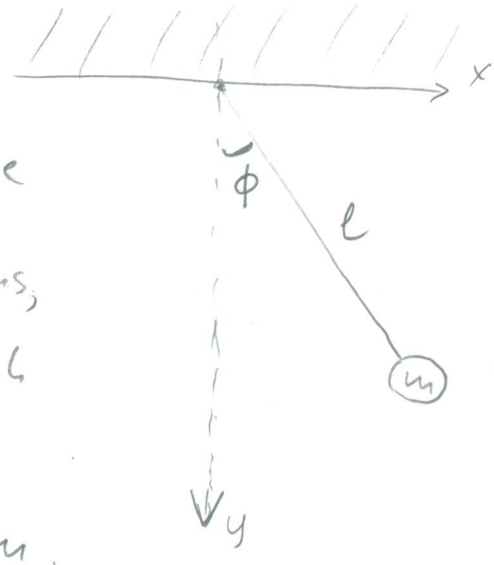
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad \longleftarrow \text{ The Lagrange equation}$$

The choice of the Lagrangian is not unique. It is defined to a full time derivative of an arbitrary function of q and t:

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dG}{dt} \qquad G = G(q, t)$$

L' will result in exactly the same equations of motion.

Example of the Lagrange formalism applied to a constrained system: plane pendulum



A bob of mass m is suspended via a massless rod. It is free to swing without friction in xy plane. The bob moves in both x and y directions, but it is constrained by the rod, such that $\sqrt{x^2+y^2}=l$. There is only one degree of freedom in this system. We could eliminate y by expressing $y=\sqrt{l^2-x^2}$. This is perfectly valid. Yet it is more convenient and natural to use parameter ϕ – the angle between the pendulum and its equilibrium position. Let us consider all quantities of interest

$$T=\frac{1}{2}mv^2=\frac{1}{2}ml^2\dot{\phi}^2$$

$$V=mgh=mgl(1-\cos\phi)$$

$$L=T-V=\frac{1}{2}ml^2\dot{\phi}^2-mgl(1-\cos\phi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}}-\frac{\partial L}{\partial\phi}=0$$

$$\frac{d}{dt}\left(ml^2\dot{\phi}\right)+mgl\sin\phi=0$$

$$ml^2\ddot{\phi}+mgl\sin\dot{\theta}=0$$

$$\ddot{\phi}+\frac{g}{l}\sin\theta=0 \quad \leftarrow$$

familiar differential equation that describes pendulum