# Velocity-dependent potentials

So far we defined conservative forces as those that can be represented as a gradient of a potential:

$$\vec{F}(\vec{r},t) = -\nabla V(\vec{r},t)$$

There may be cases, however, when forces depend not only on positions $\vec{r}$, but also no velocities $\vec{v}$. An example is the Lorentz force, which describes the interaction of a charged particle with an electromagnetic field:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \qquad \text{(SI units used)}$$

where $q$ is the charge of the particle and $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields respectively.

It turns out that Lagrange's equations can be written in their usual form even if forces depend on the velocity or the acceleration provided they can be expressed by means of a potential $V$ in the form:

$$Q_j = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right)$$

$Q_j$ is a generalized force

where $V = V(q_j, \dot{q}_j, t)$. Such potential is sometimes called generalized potential

From the previos lecture we know that the kinetic energy $T$ and the generalized forces $Q_j$ are related by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

Combining the two equations above we get

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_j}$$

or

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

if we define $L = T - V$ with $V = V(q_j, \dot{q}_j, t)$

It is possible to show that a generalized potential remains a generalized potential if we transform to some new set of (generalized coordinates), $q'_j$

By definition the generalized forces are (recall $Q_j = \sum_i' \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$)

$$Q'_j = \sum_i' Q_i \frac{\partial q_i}{\partial q'_j} \quad (*)$$

(tilde denotes quantities corresponding to the set of coordinates $\tilde{q}_i$)

We have to show that $Q'_j = -\frac{\partial \tilde{V}}{\partial q'_j} + \frac{d}{dt}\left(\frac{\partial \tilde{V}}{\partial \dot{q}'_j}\right)$

We will make use of the relation

$$\frac{\partial \dot{q}_k}{\partial \dot{q}'_j} = \frac{\partial q_k}{\partial q'_j}$$

which follows immediately from $\dot{q}_k = \frac{d}{dt} q_k = \sum_j' \frac{\partial q_k}{\partial q'_j} \dot{q}'_j + \frac{\partial q_k}{\partial t}$

Then

$$\frac{\partial \tilde{V}}{\partial q'_j} = \sum_k' \frac{\partial \tilde{V}}{\partial q_k} \frac{\partial q_k}{\partial q'_j} + \sum_k \frac{\partial \tilde{V}}{\partial \dot{q}_k} \underbrace{\frac{\partial \dot{q}_k}{\partial q'_j}}_{\frac{\partial}{\partial q'_j}\left(\sum_m' \frac{\partial q_k}{\partial q'_m} \dot{q}'_m + \frac{\partial q_k}{\partial t}\right)} + \frac{\partial \tilde{V}}{\partial t} \underbrace{\frac{\partial t}{\partial q'_j}}_{0} \quad (**)$$

Now because $Q_k = -\frac{\partial V}{\partial q_k} + \frac{d}{dt} \frac{\partial \tilde{V}}{\partial \dot{q}_k}$ we write $Q_j$ in $(*)$ as

$$Q'_j = -\sum_k' \frac{\partial \tilde{V}}{\partial q_k} \frac{\partial q_k}{\partial q'_j} + \sum_k \frac{d}{dt}\left(\frac{\partial \tilde{V}}{\partial \dot{q}_k}\right) \frac{\partial q_k}{\partial q'_j} =$$

$$= -\sum_k' \frac{\partial \tilde{V}}{\partial q_k} \frac{\partial q_k}{\partial q'_j} + \sum_k \frac{d}{dt}\left(\frac{\partial \tilde{V}}{\partial \dot{q}_k} \cdot \frac{\partial q_k}{\partial q'_j}\right) - \sum_k \frac{\partial \tilde{V}}{\partial \dot{q}_k} \frac{d}{dt}\left(\frac{\partial q_k}{\partial q'_j}\right)$$

Or, rearranging the terms in the right and left side.

$$\sum_k' \frac{\partial \tilde{V}}{\partial q_k} \frac{\partial q_k}{\partial q'_j} = -Q'_j + \sum_k \frac{d}{dt}\left(\frac{\partial \tilde{V}}{\partial \dot{q}_k} \frac{\partial q_k}{\partial q'_j}\right) - \sum_k \frac{\partial \tilde{V}}{\partial \dot{q}_k} \frac{d}{dt}\left(\frac{\partial q_k}{\partial q'_j}\right)$$

By inserting this into $(**)$ we obtain:

$$Q'_j = -\frac{\partial \tilde{V}}{\partial q'_j} + \sum_k' \frac{\partial \tilde{V}}{\partial \dot{q}_k}\left(\frac{\partial}{\partial q'_j}\left[\sum_m \frac{\partial q_k}{\partial q'_m} \dot{q}'_m + \frac{\partial q_k}{\partial t}\right]\right) + \sum_k' \frac{d}{dt}\left(\frac{\partial \tilde{V}}{\partial \dot{q}_k} \frac{\partial q_k}{\partial q'_j}\right) - \sum_k \frac{\partial \tilde{V}}{\partial \dot{q}_k} \frac{d}{dt}\left(\frac{\partial q_k}{\partial q'_j}\right)$$

The third term in the above expression is

$$\sum_k \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_k} \underbrace{\frac{\partial q_k}{\partial q'_j}}_{\overset{\shortparallel}{\frac{\partial \dot{q}_k}{\partial \dot{q}'_j}}}\right) = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}'_j}\right)$$

Thus, $Q'_j$ becomes

$$Q'_j = -\frac{\partial V}{\partial q'_j} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}'_j}\right) + \sum_k \frac{\partial V}{\partial \dot{q}_k}\left(\frac{\partial}{\partial q'_j}\left[\sum_m \frac{\partial q_k}{\partial q'_m}\dot{q}'_m + \frac{\partial q_k}{\partial t}\right]\right)$$

$$- \sum_k \frac{\partial V}{\partial \dot{q}_k}\left(\sum_m \frac{\partial}{\partial q'_m}\left[\frac{\partial q_k}{\partial q'_j}\right]\dot{q}'_m + \frac{\partial}{\partial t}\frac{\partial q_k}{\partial q'_j}\right)$$

Since $\dot{q}'_m$ does not depend explicitly on $q'_j$ the last two terms cancel out and we obtain

$$Q'_j = -\frac{\partial V}{\partial q'_j} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}'_j}\right)$$

## Example: Charged particle in EM field

If $\vec{F} = q\left[\vec{E} + \vec{v}\times\vec{B}\right]$ then $\vec{E}$ and $\vec{B}$ can be derived from the scalar and vector potential respectively:

$$\vec{E} = -\nabla\Phi - \frac{\partial\vec{A}}{\partial t} \qquad \vec{B} = \nabla\times\vec{A}$$

The force above can be derived from the following velocity-dependent potential energy

$$V = q\Phi - q\vec{A}\cdot\vec{v}$$

so that the Lagrangian reads

$$L = T - V = \frac{1}{2}mv^2 - q\Phi + q\vec{A}\cdot\vec{v}$$

Now let us consider the Lagrange equation for the x-component: $\frac{d}{dt}\frac{\partial L}{\partial v_x} = \frac{\partial L}{\partial x}$

or explicitly:

$$\frac{d}{dt}(mv_x + qA_x) = -q\frac{\partial \Phi}{\partial x} + q\frac{\partial \vec{A}}{\partial x}\cdot\vec{v}$$

Here

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial A_x}{\partial t}\frac{\partial z}{\partial t} =$$

$$= \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}v_x + \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_x}{\partial z}v_z$$

and

$$\frac{\partial \vec{A}}{\partial x}\cdot\vec{v} = \frac{\partial A_x}{\partial x}\cdot v_x + \frac{\partial A_y}{\partial x}\cdot v_y + \frac{\partial A_z}{\partial x}v_z .$$

With the latter expressions we get

$$\frac{d(mv_x)}{dt} = q\left(-\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial t}\right) + q\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)v_y - q\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)v_z$$

or

$$\frac{d(mv_x)}{dt} = eE_x + q(B_x v_y - B_y v_z) = q(\vec{E}_x + [\vec{v}\times\vec{B}]_x)$$

We can do a similar derivation for the y and z components. In total we will obtain

$$\frac{d(m\vec{v})}{dt} = q(\vec{E} + \vec{v}\times\vec{B})$$

# Dissipation function

If not all forces acting on the system are derivable from a potential, the Lagrange's equations can be written in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k$$

where $L$ contains the potential of the conservative forces, while $Q_k$ represents the nonconservative forces

In some cases we can approximate the friction force as

$$F_f = -k_x v_x$$

for a particle moving in the positive x-direction, or in three dimensions

$$F_f = -k\vec{v} \qquad (\text{if } k_x = k_y = k_z)$$

Frictional forces of this type may be derived in terms of a function, $D$, called Rayleigh's dissipation function. It is defined as

$$D = \frac{1}{2}\sum_i \left(k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2\right) \quad \text{sum over all particles}$$

Then

$$\vec{F}_f = -\nabla_{\vec{v}} D$$

The Lagrande equations can be written then as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} = 0$$

## Example: motion of a projectile in air

$V = mgz$ ← potential energy due to gravity

$$D = \frac{k}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

Lagrange's equations that follow from these L and D are:

$m\ddot{x} + k\dot{x} = 0$ $\qquad$ $m\ddot{y} + k\dot{x} = 0$ $\qquad$ $m\ddot{z} + k\dot{z} + mg = 0$