The principle of least action (Hamilton's principle)

Previously, we have derived the Lagrange equations based on D'Alembert principle, which has a differential form

$$\sum_i \left(F_i^{(a)} - \dot{\vec{p}}_i\right) \cdot \delta \vec{r}_i = 0$$

It is also possible to obtain Lagrange's equations from a principle that considers the entire motion of the system between times t_1 and t_2 and small variations of this motion from the actual motion. Such a a principle has an integral form.

For monogenic systems (systems for which all forces, except the forces of constraint, are derivable from a generalized scalar potential) the principle of least action can be stated as:

The motion of the system from time t_1 to time t_2 is such that the action integral

$$I = \int_{t_1}^{t_2} L(q_1, \dots q_n, \dot{q}_1, \dots \dot{q}_n, t)\, dt \qquad \text{where } L = T - V$$

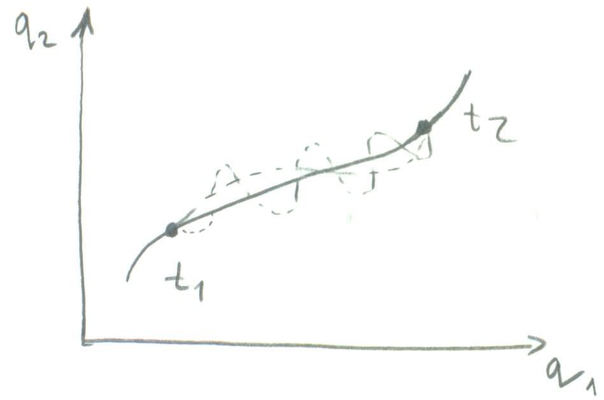
has a stationary value for the actual path of the motion. (Note: in physics literature the action is often denotes as S)

* Essentially the action integral takes the least possible value for a sufficiently short segment of the path. For the entire path the integral must have an extremum.

When we vary a finite "path" of the system, by path we understand a path in a configuration space — an n-dimensional space (n is the number of degrees of freedom) of generalized coordinates that define the position of the system.

The condition of an extremum is expressed as

$$\delta I = 0 \qquad \text{or} \qquad \delta \int_{t_1}^{t_2} L\, dt = 0$$



The principle of least action enables to construct the mechanics of monogenic systems as the basic postulate rather than Newton's laws of motion. The principle of least action is used often extended to describe the time evolution of apparently non-mechanical systems — fields. Hence it has some fundamental universality.

Before considering how Lagrange's equations emerge from the principle of least action we must learn the basics of the calculus of variation.

Elements of the calculus of variation

An illustrative example of a variational problem is the one where we need to find the shortest distance between two points (straight line).

In rectangular coordinates the straight line is uniquely determined by fixing two of its points and it can also be described by a linear equation (in coordinates x and y). It can further be described by the differential equation

$$\frac{d^2y}{dx^2} = 0$$

combined with the condition that the values of the desired function $y(x)$ for $x = x_1$ and $x = x_2$ are given numbers.

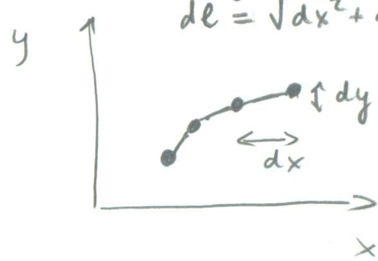
The straight line, however, can also be described as the shortest connection between two points, i.e.

$$\int ds = \min$$

One can imagine that the two given points are being connected by all possible curves and among these curves that curve be selected, which yields the minimum value of the integral. An important property of such a formulation is that it is independent of of the choice of particular coordinates.

After introducing rectangular coordinates x and y (as an example), the problem is to look for a function $y(x)$ for which $y(x_1)$ and $y(x_2)$ have given values and the integral

$$dy = y'dx$$
$$dl = \sqrt{dx^2 + dy^2}$$



$$I = \int_{x_1}^{x_2} \sqrt{1 + [y'(x)]^2}\, dx$$

takes a minimum value

This problem has some similarity with the search for the minimum of a given function. There one considers a small change of x and forms

$$df = f'(x)\, dx$$

If $f'(x) \neq 0$ then $f(x)$ can increase or decrease for small change of x, and thus, there is no minimum at point x. A necessary condition for a minimum is therefore $f'(x) = 0$. This condition is not sufficient, however. It is also fulfilled for a maximum.

In our problem above we do not have to change a variable. Instead we need to change (or as they say vary) a function $y(x)$.

We replace $y(x)$ by a "neighboring" function

$$y_0(x) + \varepsilon \eta(x) \quad (\text{or } y_0(x) + \delta y)$$

of the desired function $y_0(x)$, where we will assume ε is arbitrarily small. We must have $\eta_1(x_1) = \eta(x_2) = 0$. $y'(x)$ is then replaced by $y_0' + \varepsilon\eta'$ and instead of the integrand $\sqrt{1+y'^2}$ we obtain the Taylor series expansion into powers of ε:

$$\sqrt{1+(y_0'+\varepsilon\eta')^2} = \sqrt{1+y_0'^2} + \varepsilon \frac{y_0'}{\sqrt{1+y_0'^2}}\eta' + \varepsilon^2(\ldots) + \ldots$$

With that we have

$$I(\varepsilon) = \int_{x_1}^{x_2} \sqrt{1+(y_0'+\varepsilon\eta')^2}\,dx \approx \int_{x_1}^{x_2} \sqrt{1+y_0'^2}\,dx + \varepsilon \int_{x_1}^{x_2} \frac{y_0'}{\sqrt{1+y_0'^2}}\eta'\,dx$$

If the integral in the second term does not vanish then the integral $\int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx$ can increase or decrease by changing the function $y(x)$, depending on the sign of ε. For a minimum to exist we must require

$$\int_{x_1}^{x_2} \frac{y_0'}{\sqrt{1+y_0'^2}}\eta'\,dx = 0$$

for any function $\eta(x)$ that vanishes at x_1 and x_2

Now let us do the integration by parts:

$$\int_{x_1}^{x_2} \frac{y_0'}{\sqrt{1+y_0^2}}\eta\,dx = \underbrace{\left[\frac{y_0'}{\sqrt{1+y_0'^2}}\eta\right]_{x_1}^{x_2}}_{=0,\ \text{because } \eta(x_1)=\eta(x_2)=0} - \int_{x_1}^{x_2} \eta \frac{d}{dx}\left(\frac{y_0'}{\sqrt{1+y_0'^2}}\right)dx = 0$$

The las term in the above equation becomes zero only if

$$\frac{d}{dx}\left(\frac{y_0'}{\sqrt{1+y_0'^2}}\right) = 0$$

for if this were not satisfied everywhere, we could choose $\eta(x)$ in such a way that it is always positive where $\frac{d}{dx}\left(\frac{y_0'}{\sqrt{1+y_0'^2}}\right)$ is positive and choose it negative where this expression is negative. Therefore it follows that

$$y_0' = \text{constant}$$

and

$$y_0'' = 0$$

which provides a differential equation for a straight line. Our calculation has replaced the requirement that a definite integral be minimized by a function, by a differential equation for this quation