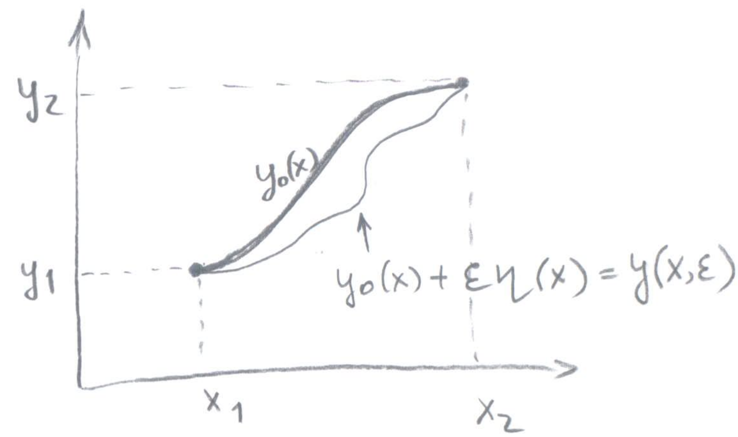
General discussion of variational principles

Let us consider the following one-dimensional problem. Given an integrable function $F(y, y', x)$ defined on a path $y = y(x)$ between two values x_1 and x_2, where $y' = \frac{dy}{dx}$, we wish to find a particular path such as the line integral

$$I = \int_{x_1}^{x_2} F(y, y', x)\, dx$$

takes an extremum value (e.g. I has a stationary value relative to path differing infinitesimally from the $y(x)$ which realizes the extremum)

Our problem is transformed into an elementary extremum value problem by covering the ensemble of all physically meaningful paths by a parametric representation:



$$y(x, \varepsilon) = y_0(x) + \varepsilon \eta(x)$$

where ε is a parameter constant for every path, $\eta(x)$ is an arbitrary differentiable funtion that vanishes at x_1 and x_2: $\eta(x_1) = \eta(x_2) = 0$.

The desired curve is given by $y_0(x) = y(x, 0)$

For any such parametric family of curves I is also a funtion of ε

$$I(\varepsilon) = \int_{x_1}^{x_2} F(y(x,\varepsilon), y'(x,\varepsilon), x)\, dx$$

The condition for an extremum value of integral I is then

$$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

The differentiation under the integral symbol (allowed if F is continuously differentiable with respect to ε) yields

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\frac{\partial y}{\partial \varepsilon} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \varepsilon} + \frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial \varepsilon}}_{0} \right) dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' \right) dx$$

The second term can be integrated by parts

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\frac{\partial \eta}{\partial x}\, dx = \underbrace{\left.\left[\frac{\partial F}{\partial y'}\eta\right]\right|_{x_1}^{x_2}}_{0,\ \text{because } \eta(x_1)=\eta(x_2)=0} - \int_{x_1}^{x_2} \left(\frac{d}{dx}\frac{\partial F}{\partial y'} \right) \eta\, dx$$

Then the extremum condition becomes

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right) \eta\, dx = 0$$

The "fundamental lemma" of the calculus of variations says if

$$\int_{x_1}^{x_2} G(x)\, \eta(x)\, dx = 0$$

for any arbitrary function $\eta(x)$ continuous through the second derivative then $G(x)$ must vanish. Therefore we can write

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) = 0 \quad \leftarrow \text{Euler-Lagrange equation}$$

The solution of the Euler-Lagrange equation, a differential equation of second order, together with boundary conditions, yields the path sought.

Let us define the variation of a function $y(x,\varepsilon)$ as

$$\delta y = y(x, d\varepsilon) - y(x,0) = \frac{\partial y}{\partial \varepsilon}\bigg|_{\varepsilon=0} \cdot d\varepsilon \qquad d\varepsilon \to 0$$

Then the variational problem can be formulated as

$$\delta \int_{x_1}^{x_2} F(y(x), y'(x), x)\, dx = 0$$

Function F can also include constraints by means of Lagrange multipliers.

Generalization to many independent variables and derivation of Lagrange's equations

A variation of the integral I in case of multiple independent variables $y_i(x)$

$$\delta I = \delta \int_{x_1}^{x_2} F(y_1(x), \dots, y_n(x); y_1'(x), \dots y_n'(x), x)\, dx$$

is obtained by considering I as a function of a parameter that labels sets of curves

$$y_1(x,\varepsilon) = y_1(x,0) + \varepsilon \eta_1(x)$$

$$\vdots$$

$$y_n(x,\varepsilon) = y_n(x,0) + \varepsilon \eta_n(x)$$

where $\eta_1 \dots \eta_n$ are independent arbitrary functions that vanish at x_1 and x_2

$$\delta I = \frac{\partial I}{\partial \varepsilon} d\varepsilon = \int_{x_1}^{x_2} \sum_{i=1}^{n} \left(\frac{\partial F}{\partial y_i} \frac{\partial y_i}{\partial \varepsilon} d\varepsilon + \frac{\partial F}{\partial y_i'} \frac{\partial y_i'}{\partial \varepsilon} d\varepsilon \right) dx$$

Again we integrate by parts the second term

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y_i'} \frac{\partial^2 y_i}{\partial \varepsilon \partial x} dx = \underbrace{\frac{\partial F}{\partial y_i'} \frac{\partial y_i}{\partial \varepsilon} \Big|_{x_1}^{x_2}}_{0} - \int_{x_1}^{x_2} \frac{\partial y_i}{\partial \varepsilon} \frac{d}{dx} \left(\frac{\partial F}{\partial y_i'} \right) dx$$

Then

$$\delta I = \int_{x_1}^{x_2} \sum_{i=1}^{n} \left(\frac{\partial F}{\partial y_i} - \frac{d}{dx} \frac{\partial F}{\partial y_i'} \right) \delta y_i \, dx \qquad \text{where } \delta y_i = \frac{\partial y_i}{\partial \varepsilon}\Big|_{\varepsilon=0} \cdot d\varepsilon$$

By the extersion of the "fundamental lemma" the condition of extrema requires that all coefficients by δy_i vanish:

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial F}{\partial y_i'} \right) = 0 \qquad i = 1, \dots n$$

If we now change notations and replace

$$F \to L \qquad y_i \to q_i \qquad y_i' \to \dot{q}_i \qquad x \to t$$

then the principle of least action $\delta I = 0$

with $I = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) \, dt$

yields

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, \dots, n$$

under assumption that q_i are all independent, which requires that the constraints be holonomic