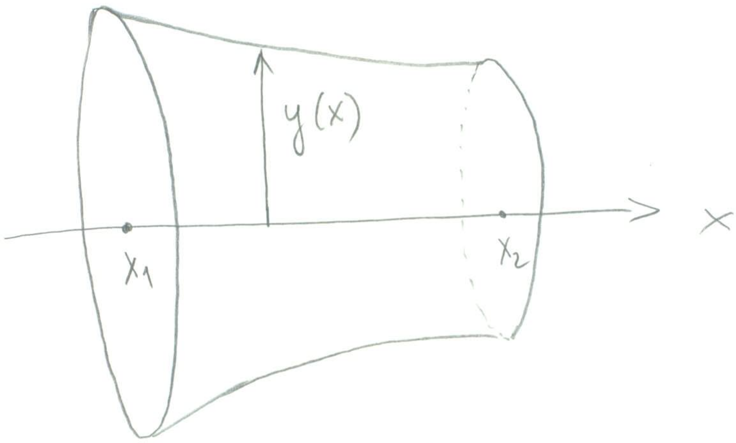
Further discussion of variational problems

To solidify our experience with the variational problems let us consider another illustrative example. — the problem of a soap film supported by a pair of coaxial ring.



It is known that the free energy of the soap film is equal to twice (once for each liquid-air interface) the surface tension σ of the soap solution times the area of the film. The film can therefore minimize its free energy by minimizing its area. Due to the axial symmetry the minimal surface will be a surface of revolution about the x-axis. We need, therefore, to find such a profile $y(x)$ that makes the area of the surface of revolution the least among all such surfaces bounded by the circles of radii $y(x_1) = y_1$ and $y(x_2) = y_2$.

$$\text{Area} = I[y(x)] = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx$$

The minimal surface corresponds to a stationary point of functional I. As we have learned by now, the stationary points can be found by solving the Euler-Lagrange equation, $\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$

In our case $F(y, y', x) = y\sqrt{1+y'^2}$

The partial derivatives are then

$$\frac{\partial F}{\partial y} = 2\pi\sqrt{1+y'^2} \qquad \frac{\partial F}{\partial y'} = \frac{2\pi y y'}{\sqrt{1+y'^2}}$$

and the Euler-Lagrange equation becomes

$$\sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) = 0$$

$$\sqrt{1+y'^2} - \frac{(y')^2}{\sqrt{1+y'^2}} - \frac{yy''}{\sqrt{1+y'^2}} + \frac{y(y')^2y''}{(1+y'^2)^{3/2}} = 0$$

or, after simplification,

$$\frac{1}{\sqrt{1+y'^2}} - \frac{yy''}{(1+y'^2)^{3/2}} = 0$$

If we multiply both sides by y' then

$$0 = \frac{y'}{\sqrt{1+y'^2}} - \frac{yy'y''}{(1+y'^2)^{3/2}} = \frac{d}{dx}\left(\frac{y}{\sqrt{1+y'^2}}\right)$$

From which it follows that

$$\frac{y}{\sqrt{1+y'^2}} = c_1 \qquad \text{or} \qquad y' = \sqrt{\frac{y^2}{c_1^2} - 1}$$

$$\int \frac{dy}{\sqrt{\frac{y^2}{c_1^2}-1}} = \int dx$$

here we can substitute $y = c_1 \cosh z$ and obtain

$$\int dx = c_1 \int dz \implies x + c_2 = c_1 z$$

This leads to $y = c_1 \cosh \frac{x+c_2}{c_1}$

Constants c_1 and c_2 are determined by using $y(x_1) = y_1$ $y(x_2) = y_2$

Extending the least action principle to systems with constraints

Using the method of Lagrange multipliers we can account for constraints in the action integral

$$I = \int_{t_1}^{t_2} L\, dt \longrightarrow I = \int_{t_1}^{t_2} \Big(L + \sum_{j=1}^{m} \lambda_j g_j\Big)\, dt$$

In this case we do not need to worry about the independence of all q_i (generalized coordinates). Instead we are allowed to vary all q_i's and all λ_j's independently.

The variations of the λ_j's give the m constraint equations

$$g_j(q_1, \dots q_n, t) = 0 \qquad j = 1 \dots m$$

Here, as before, we only consider holonomic constraints.

The variations of q_i's give

$$\delta I = \int_{t_1}^{t_2} dt \left(\sum_{i=1}^{n} \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_j} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial q_i} \right] \delta q_i \right) = 0$$

Since the δq_i are not all independent, we choose λ_j's so that m of the equations are satisfied for arbitrary δq_i and then choose the variations of the δq_i in the remaining $n-m$ equations independently. Then we obtain

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial q_i} = 0 \qquad i = 1 \dots m$$

The equality follows from the choice of λ_j's

We also have the same equations for $j = m+1, \dots, n$ where the equality follows from the virtual variations of the δq_i

We can also understand this by writing

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum \lambda_j \frac{\partial g_j}{\partial q_i} = Q_i$$

where Q_i are generalized forces