Integrating the equations of motion in 1D

Let us consider the motion of a system that has only one degree of freedom. The most general for of the Lagrangian of such a system is

$$L = \frac{1}{2} f(q)\dot{q}^2 - V(q)$$

where $f(q)$ is some function of the generalized coordinate q.

A peculiar feature of the equations of motion corresponding to these Lagrangians is that they can be integrated in a general form. Moreover, it is not even necessary to write down the equation of motion — we can start from the first integral, which gives the law of conservation of energy:

$$\frac{1}{2} f(q)\dot{q}^2 + V(q) = E$$

This first-order differential equation can be integrated immediately

$$\frac{dq}{dt} = \sqrt{\frac{2(E-V(q))}{f(q)}} \qquad t = \int \sqrt{\frac{f(q)}{2(E-V(q))}}\,dq + \text{const.}$$

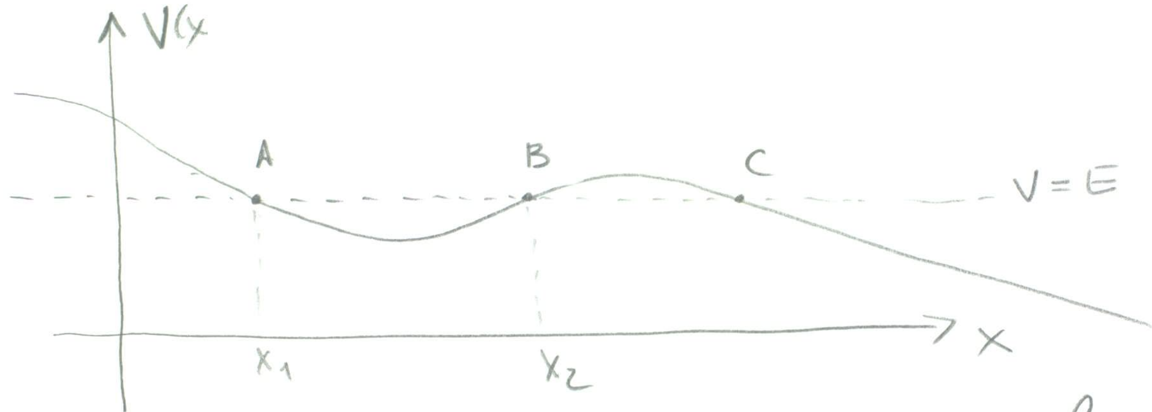
or, if we work in Cartesian coordinates where $f(q) = m$

$$t = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E-V(x)}} + \text{const}$$

The two arbitrary constants in the solution of the equations of motion are here represented by the total energy and the constant of integration.

Since the kinetic energy is positive, the total energy E always exceeds the potential energy. Hence the motion is possible in those regions of space where $V(x) < E$.

The points at which the potential energy equals the total energy, $V(x) = E$, give the limits of the motion. They are called turning points (because the velocity changes direction there)



If the region of the motion is bounded by two such points then the motion takes place in a finite region of space, and is said to be finite. If the region of the motion is limited on one side only, or on neither, then the motion is infinite.

A finite motion in 1D is oscillatory (i.e. periodic). The period of oscillation, τ, is

$$\tau(E) = \sqrt{2m} \int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{E - V(x)}}$$

where x_1 and x_2 are roots of the equation $V(x) = E$. The factor of 2 appears in the numerator (not denominator) because in one period the particle passes from x_1 to x_2 and then back from x_2 to x_1.

Motion in 3D. Reduction of a two-body problem to the equivalent one-body problem

A problem of two interacting particles can be transformed into the equivalent problem of one particle moving in the field of the other.

Let us consider a monogenic system of two particles with masses m_1 and m_2 and forces arise due to an interaction potential V. We will assume first that $V = V(\vec{r}, \dot{\vec{r}}, \ddot{\vec{r}}, \dots)$ where $\vec{r} = \vec{r}_2 - \vec{r}_1$. This system has six degrees of freedom. Instead of $\vec{r}_1$ and $\vec{r}_2$ we can choose

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \quad \leftarrow \text{center of mass}$$

and

$$\vec{r} = \vec{r}_2 - \vec{r}_1$$

to be the six generalized coordinates. The Lagrangian

$$L = T(\dot{\vec{R}}, \dot{\vec{r}}) - V(\vec{r}, \dot{\vec{r}}, \ddot{\vec{r}}, \dots)$$

contains the kinetic energy that can be written as

$$T = \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + T' \qquad \text{where } T' = \frac{1}{2}m_1\dot{\vec{r}}_1'^2 + \frac{1}{2}m_2\dot{\vec{r}}_2'^2$$

and

$$\vec{r}_1' = -\frac{m_2}{m_1+m_2}\vec{r} \qquad \vec{r}_2' = \frac{m_1}{m_1+m_2}\vec{r}$$

are the coordinates of the two particles relative to the center of mass. T' takes on the form

$$T' = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2$$

while the total Lagrangian becomes

$$L = \frac{m_1+m_2}{2}\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2 - V(\vec{r},\dot{\vec{r}},\ddot{\vec{r}},\dots)$$

The three coordinates $\vec{R}$ are now cyclic. From that fact it follows that the center of mass is either at rest or moving uniformly. The equations of motion for $\vec{r}$ will not contain terms involving $\vec{R}$ or $\dot{\vec{R}}$. Hence we can drop the term $\frac{m_1+m_2}{2}\dot{\vec{R}}^2$ from consideration. The remaining part of the Lagrangian looks the same as one for a fixed center of force with a single particle at a distance $\vec{r}$ from it, but with a mass given by

$$\mu = \frac{m_1 m_2}{m_1+m_2} \quad \leftarrow \text{reduced mass}$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

The equations of motion for a particle in a central field

Let us now restrict ourselves to the case of a central potential $V(r)$. Since V is a function of $|\vec{r}|$ only, the force is along the $\vec{r}$ vector. Since the problem has spherical symmetry, no rotation about a fixed axis will effect the solution. Angle coordinates representing such a rotation must be cyclic then.

The total angular momentum $\vec{L} = \vec{r}\times\vec{p}$ is conserved. $\vec{r}$ is always perpendicular to the fixed direction of $\vec{L}$. This can only be true if $\vec{r}$ lies in a plane that

that is perpendicular to $\vec{L}$

Now we introduce the spherical coordinates for our particle — r, θ, and ψ. We choose the polar axis in the direction of $\vec{L}$, so coordinate ψ has a constant value of $\pi/2$. We can drop it from subsequent consideration.

Expressed in polar coordinates the Lagrangian reads

$$L = T - V = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - V(r)$$

Here θ is a cyclic coordinate. Its canonical momentum is the angular momentum of the system:

$$p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

One of the two equations of motion is then

$$\dot{p}_\theta = \frac{d}{dt}\left(m r^2 \dot{\theta} \right) = 0$$

an the first integral is

$$m r^2 \dot{\theta} = \ell$$

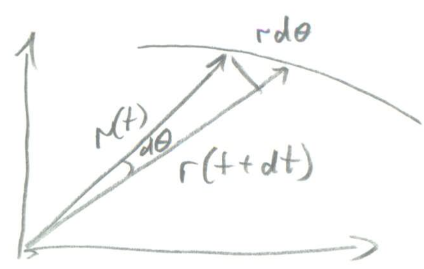
ℓ is the value of the angular momentum

An equivalent form of the above integral is a statement

$$\frac{d}{dt}\left(\frac{1}{2} r^2 \dot{\theta} \right) = 0$$

if $dA \equiv \frac{1}{2} r (r\, d\theta)$ is the infinitesimal area swept by the radius vector then

$$\underbrace{\frac{dA}{dt}}_{\text{areal velocity}} = \underbrace{\frac{1}{2} r^2 \frac{d\theta}{dt}}_{\text{const}}$$



The Lagrange equation for coordinate r is

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

or if we define $f(r) \equiv -\frac{\partial V}{\partial r}$

$$m\ddot{r} - mr\dot{\theta}^2 = f(r)$$

Using the first integral $mr^2\dot{\theta} = e$, $\dot{\theta}$ can be eliminated, yielding

$$m\ddot{r} - \frac{e^2}{mr^3} = f(r)$$

There is another first integral of motion, E, since the force field $V(r)$ is conservative

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

It can be derived from the equations of motion

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0 \quad \text{and} \quad m\ddot{r} - \frac{e^2}{mr^3} = f(r)$$

The last equation can be written as

$$m\ddot{r} = -\frac{d}{dr}\left(V + \frac{1}{2}\frac{e^2}{mr^2}\right)$$

if we multiply both sides by $\dot{r}$ it becomes

$$m\ddot{r}\dot{r} = \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right)$$

if then $g(r)$ is defined as $\frac{d}{dt}g(r) = \frac{dg}{dr}\frac{dr}{dt}$

then

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = -\frac{d}{dt}\left(V + \frac{1}{2}\frac{e^2}{mr^2}\right)$$

or

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{e^2}{mr^2} + V\right) = 0$$

From this it follows that

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{mr^2} + V = \text{const}$$

Since

$$\frac{1}{2}\frac{\ell^2}{mr^2} = \frac{1}{2mr^2}m^2r^4\dot{\theta}^4 = \frac{mr^2\dot{\theta}^2}{2}$$

the previous equation reduces to

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$$

The remaining integrations can be proceed with as follows. Solving for $\dot{r}$ we get

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)} \quad \text{or} \quad dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)}}$$

Let $r(t=0) = r_0$. Then

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)}}$$

$t(r)$ can be iverted (at least formally) and give us $r(t)$. Then the solution for θ can be easily obtained:

$$\frac{d\theta}{dt} = \frac{\ell\, dt}{mr^2}$$

and

$$\theta = \ell\int_0^t \frac{dt}{mr^2(t)} + \theta_0 \qquad \text{where} \quad \theta_0 \equiv \theta(t=0)$$