The virial theorem.

If the motion takes place in a finite region of space we can establish a simple and useful relation between the time average values of the kinetic and potential energies, known as the virial theorem. This theorem is statistical in nature.

Let us consider a general system of particles with coordinates $\vec{r}_i$ and applied forces $\vec{F}_i$ (including the forces of constraint). For each particle

$$\dot{\vec{p}}_i = \vec{F}_i$$

If we introduce the quantity $G = \sum_i \vec{p}_i \cdot \vec{r}_i$, its total time derivative is

$$\frac{dG}{dt} = \underbrace{\sum_i{}' \dot{\vec{r}}_i \cdot \vec{p}_i}_{\sum_i m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = \sum_i m_i v_i^2 = 2T} + \underbrace{\sum_i{}' \dot{\vec{p}}_i \cdot \vec{r}_i}_{\sum_i \vec{F}_i \cdot \vec{r}_i}$$

Then

$$\frac{d}{dt} \sum_i{}' \vec{p}_i \cdot \vec{r}_i = 2T + \sum_i \vec{F}_i \cdot \vec{r}$$

Time averaging the latter equation over a sufficiently long time interval τ yields

$$\frac{1}{\tau} \int_0^\tau \frac{dG}{dt}\, dt \equiv \underbrace{\overline{\frac{dG}{dt}}}_{\frac{1}{\tau}\left[G(\tau) - G(0)\right]} = \overline{2T} + \overline{\sum_i{}' \vec{F}_i \cdot \vec{r}_i}$$

If the motion is periodic and τ is chosen to be the period then $G(\tau) - G(0) = 0$. But even if the motion is not exactly periodic (just finite) and $\tau \to \infty$ the left-hand side vanishes. Then

$$\overline{T} = \underbrace{-\frac{1}{2}\overline{\sum_i \vec{F}_i \cdot \vec{r}_i}}_{\text{virial of Clausius}} \qquad \leftarrow \text{virial theorem}$$

When the forces are derivable from a potential the theorem becomes

$$\overline{T} = \frac{1}{2}\overline{\sum_i \frac{\partial V}{\partial \vec{r}} \cdot \vec{r}}$$

In the case of a single particle moving in a central field

$$\overline{T} = \frac{1}{2}\overline{\frac{\partial V}{\partial r} r}$$

The virial theorem is particularly useful V is a power-law function, $V = \alpha r^{n+1}$. In this case

$$\frac{\partial V}{\partial r} r = (n+1) V$$

and

$$\overline{T} = \frac{n+1}{2}\overline{V}$$

In particular, for a harmonic oscillator potential $(n=1)$ we get $\overline{T} = \overline{V}$, while for the Coulomb potential $\overline{T} = -\frac{1}{2}\overline{V}$

It should be noted that the applications of the virial theorem are not limited to classical mechanics. It finds its way to quantum mechanics as well