Equation for the orbit in a central field

While the usual formal task of mechanis is to find coordinates (such as r and θ) as functions of time, it is often of great interest to determine the equation of the orbit, i.e. the dependence of r on θ, where the dependence on t is eliminated. For a central force problem this elimination is particularly simple. Indeed, as we have determined in a previos lecture

$$mr^2\frac{d\theta}{dt} = \ell \quad (*) \implies \ell\, dt = mr^2\, d\theta$$

and

$$\frac{d}{dt} = \frac{\ell}{mr^2}\frac{d}{d\theta}$$

if we substitute this relation for $\frac{d}{dt}$ into another equation we derived in a previous lecture, namely

$$m\ddot{r} - \frac{\ell^2}{mr^3} = f(r) \quad , \text{ where } \quad f(r) \equiv -\frac{\partial V}{\partial r}.$$

then we obtain

$$\frac{1}{r^2}\frac{d}{d\theta}\left(\frac{\ell^2}{mr^2}\frac{dr}{d\theta}\right) - \frac{\ell^2}{mr^3} = f(r)$$

Let us define $u \equiv 1/r$. With respect to u the above equation reads as

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{\ell^2}\frac{d}{du}V\left(\frac{1}{u}\right)$$

For any particular force law, the actual equation of the orbit can be obtained by eliminating t from yet another formula we derived previously:

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)}}$$

by means of $(*)$ above:

$$d\theta = \frac{\ell\, dr}{mr^2\sqrt{\frac{2}{m}\left(E - V(r) - \frac{\ell^2}{2mr^2}\right)}}$$

and then

$$\theta = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}} + \theta_0$$

In the case if the variable is u (not r) it becomes

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - u^2}}$$

The integral is not always can be expressed in terms of elementary functions. Even for $V = \alpha r^{n+1}$ (the power law potential) it is possible only for some selected values on n $(1, -2, -3)$

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{\ell^2} - \frac{2ma}{\ell^2} u^{-n-1} - u^2}}$$

The Kepler problem

Now let us consider a specific case when $n=-2$, which corresponds to the force proportional to the inverse square of the distance.

$$V(r) = -\frac{K}{r} \qquad f(r) = -\frac{K}{r^2}$$

Using the above formula we have

$$\theta = \theta' - \int \frac{du}{\sqrt{\frac{2mE}{\ell^2} + \frac{2mKu}{\ell^2} - u^2}}$$

here we have an indefinite integral and a constant of integration θ'

The indefinite integral is of the form

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \arccos\left(-\frac{\beta + 2\gamma x}{\sqrt{q}}\right) \qquad q = \beta^2 - 4\alpha\gamma$$

In our case $\alpha = \frac{2mE}{e^2}$, $\beta = \frac{2mk}{e^2}$, $\gamma = -1$. Then

$$q = \left(\frac{2mk}{e^2}\right)^2 \left(1 + \frac{2Ee^2}{mk^2}\right)$$

and we get

$$\theta = \theta' - \arccos \frac{\frac{e^2 u}{mk} - 1}{\sqrt{1 + \frac{2Ee^2}{mk^2}}}$$

Solving for r yields

$$\frac{1}{r} = \frac{mk}{e^2}\left(1 + \sqrt{1 + \frac{2Ee^2}{mk^2}} \cos(\theta - \theta')\right)$$

This is an equation in the form

$$\frac{1}{r} = C\left(1 + \varepsilon \cos(\theta - \theta')\right)$$

where e is called eccentricity. In our case

$$\varepsilon = \sqrt{1 + \frac{2Ee^2}{mk^2}}$$

The shape of the orbit depends on the value of ε

$\varepsilon > 1$, $E > 0$ - hyperbolic orbit

$\varepsilon = 1$, $E = 0$ - parabolic orbit

$\varepsilon < 1$, $E < 0$ - ellipse

$\varepsilon = 0$, $E = -\frac{mk^2}{2e^2}$ - circle

In the case of an elliptic orbit the major axis depends only on the energy. By definition the radial velocity is zero at the point of semimajor axes. The conservation of energy then implies that r_1 and r_2 are the roots of the previously derived equation

$$E - \frac{e^2}{2mr^2} + \frac{k}{r} = 0 \qquad \left[\text{Equation } \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{e^2}{mr^2} + V = E\right]$$

or

$$r^2 + \frac{k}{E} r - \frac{\ell^2}{2mE} = 0 \quad \Longrightarrow \quad a = \frac{r_1 + r_2}{2} = -\frac{k}{2E}$$

In terms of the semimajor axis the eccentricity can be written as

$$\varepsilon = \sqrt{1 - \frac{\ell^2}{mka}}$$

or

$$\frac{\ell^2}{mk} = a(1-\varepsilon^2)$$

Then the elliptic orbit becomes

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos(\theta-\theta')}$$

Useful terms:

perihelion $\quad r = a(1-\varepsilon)$

aphelion $\quad r = a(1+\varepsilon)$