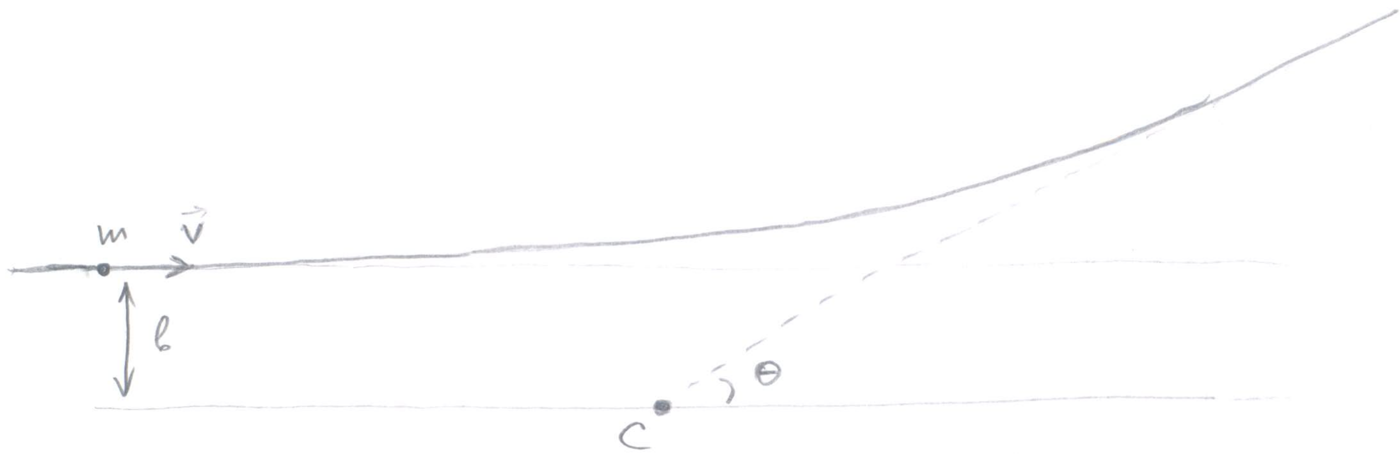
Scattering in central field

Let us consider the case of an infinite motion, when the total energy of a particle moving in a central field is such that this particle is scattered off. This situation is depicted below.



The particle of mass m approaches the vicinity of the scattering center C (which can be another and very heavy particle) in such a way that if there were no force acting between particles it would pass point C with a distance of closest approach b. This quantity, b, is called the impact parameter. If the velocity of the particle at infinity is v then the impact parameter determines the angular momentum l

$$l = mvb$$

We may also express v in terms of the incident kinetic energy T_i (which is equal to E if $V(r=\infty)=0$)

$$l = b\sqrt{2mT_i} = b\sqrt{2mE}$$

Evidently, for a given energy T_i the angular momentum and scattering angle θ is uniquely specified by the impact parameter if the interaction (force law) is known.

Due to spherical symmetry it is also evident that we may restrict ourselves to considering the dependence of angle θ only (not asimutal angle ϕ). The trajectories with the same b value and different angle ϕ will form an axially symmetric shape.

Let us now consider the distribution of scattering angles that result from scattering with various impact parameters. If we have a beam of incident particles we have define the intensity (or flux density) I of the incident particles as the number of particles passing in a unit of time through a unit area normal to the direction of the beam

Let us now define a differential scattering cross section $\sigma(\theta)$

$$\sigma(\theta)\, d\Omega = \frac{\#\text{ particles scattered into solid angle } d\Omega \text{ per unit time}}{\text{incident intensity}}$$

or

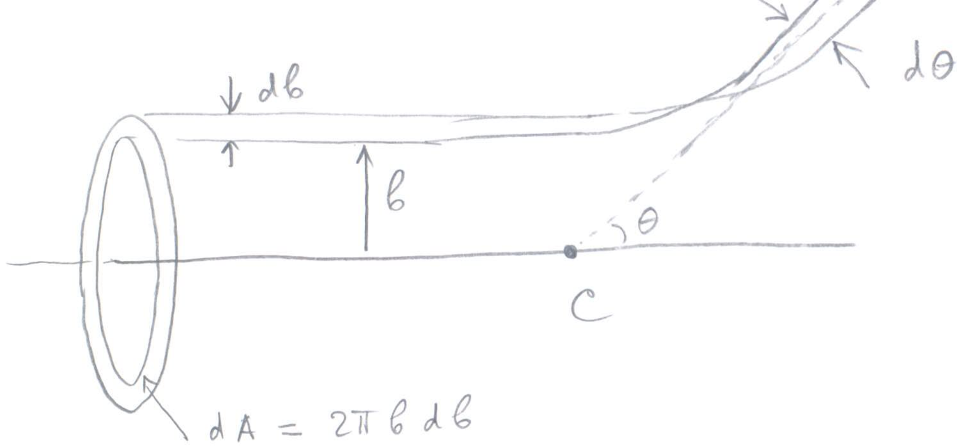
$$\sigma(\theta)\, d\Omega = \frac{dN}{I}$$

or, alternatively

$$\sigma(\theta) = \frac{1}{I}\frac{dN}{d\Omega}$$

$\sigma(\theta)$ has the dimensions of area (per steradian) which gives rise to the term cross section

For spherically symmetric scattering ϕ can be integrated over: $\quad d\Omega = 2\pi \sin\theta \, d\theta$



The number of particles with impact parameters within a range db at distance b must correspond to the number of particles scattered into the angular range $d\theta$ at angle θ. Therefore,

$$I \cdot 2\pi b\, db = \pm I\, \sigma(\theta)\, 2\pi \sin\theta\, d\theta$$

the $\pm$ sign is there because the force law may be either attractive or repulsive, so $\frac{db}{d\theta}$ may be either positive or negative. With the above formula we see that

$$\sigma(\theta) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right|$$

In the previous lecture we obtaine the relationship for angle θ as a function of r

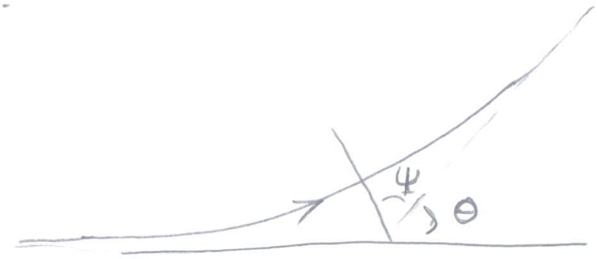
$$\theta = \int_{r_0}^{r} \frac{dr'}{r'^2 \sqrt{\frac{2mE}{\ell^2} - \frac{2mV(r')}{\ell^2} - \frac{1}{r'^2}}} + \theta_0$$

For a scattered particle the change in angle

$$\Delta\theta = \int_{r_{min}}^{r_{max}} \frac{dr}{r^2 \sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}}$$

The motion of particle is symmetric (in a central force field) about the point of closest approach to the scattering center. This follows from the time reversal symmertry.

Then



$$\theta = \pi - 2\psi$$

where ψ is the angle between the direction of the incoming asymptote and the periapsis (closest approach). Then

$$\psi = \int_{r_{min}}^{\infty} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}}$$

If we express ℓ in terms of the impact parameter, b, it becomes

$$\theta(b) = \pi - 2\int_{r_{min}}^{\infty} \frac{b\,dr}{r\sqrt{r^2\left(1-\frac{V}{E}\right) - b^2}}$$

In case if we want to use variable $u = 1/r$ instead it reads

$$\theta(b) = \pi - 2\int_{0}^{u_{max}} \frac{b\,du}{\sqrt{1 - \frac{V(u)}{E} - b^2u^2}}$$

Rutherford scattering

One of the most important problems that makes use of the above formula is the scattering of charged particles in a Coulomb field

$$V(r) = \frac{\alpha}{r} \quad \text{where} \quad \alpha = \frac{q_1 q_2}{4\pi\varepsilon_0} \leftarrow \text{two charged particles}$$

$$\psi = \int_{r_{min}}^{\infty} \frac{b/r\,dr}{\sqrt{r^2 - \frac{\alpha r}{E} - b^2}} = \arccos\frac{\gamma/b}{\sqrt{1+(\gamma/b)^2}}$$

(here r_{min} is found by solving $E = V + \frac{\ell^2}{2mr^2}$

above we defined $\gamma = \frac{\alpha}{2E}$

The equation for ψ can be rewritten as

$$b^2 = \gamma^2 \tan\psi \qquad \left(\text{because } \arccos\frac{x}{\sqrt{1+x^2}} = \arctan\frac{1}{x}\right.$$

At the same time $\psi = \frac{\pi}{2} - \frac{\theta}{2}$, so

$$b = \gamma \cot\left(\frac{\theta}{2}\right)$$

and

$$\frac{db}{d\theta} = -\frac{\gamma}{2}\frac{1}{\sin^2\frac{\theta}{2}}$$

Then the equation $\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$ becomes

$$\sigma(\theta) = \frac{\gamma^2}{2}\frac{\cot\left(\frac{\theta}{2}\right)}{\sin\theta\,\sin^2\left(\frac{\theta}{2}\right)}$$

Now since $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ we get

$$\sigma(\theta) = \frac{\gamma^2}{4}\frac{1}{\sin^4\frac{\theta}{2}} = \frac{\alpha^2}{(4E)^2}\frac{1}{\sin^4\frac{\theta}{2}}$$

which is a well-known Rutherford scattering formula.

The preceeding derivations concerned the differential scattering cross section. If it is desired to know the probability that any interaction whatsoever will take place, it is necessary to integrate $\sigma(\theta)$ over all possible scattering angles. The resulting quantity is called the total scattering cross section

$$\sigma_{tot} = \int_{4\pi} \sigma(\theta)\, d\Omega = 2\pi \int_0^{\pi} \sigma(\theta)\sin\theta\, d\theta$$

If we attempt to calculate σ_{tot} for the case of

Rutherfor scattering we find the the integral diverge and the result is infinite. This is because the Coulomb potential, $\frac{\alpha}{r}$, falls off so slowly that as b becomes infinitely large the decrease in scattering angle is too slow.