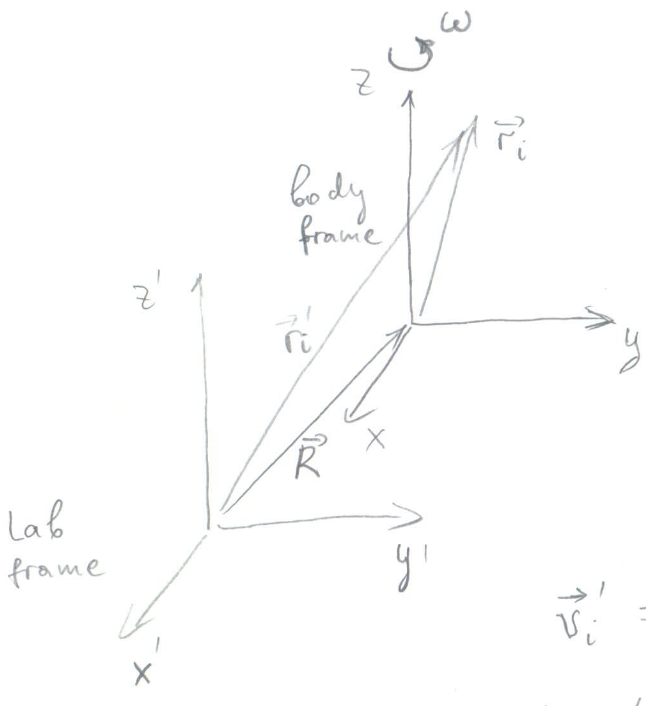
Review of basic results for rotational motion of rigid bodies

A rigid body is a collection of N particles with the property that the distances between any of its constituent particles are fixed. While the arbitrary system requires $3N$ coordinates to specify its configuration, the rigid body requires only six such coordinates.

Since all interparticle distances in a rigid body are fixed, the internal potential energy, $V^{int} = \sum_{ij}' V_{ij}(\vec{r}_{ij})$, is a constant and can be dropped from consideration.

Let us consider a rotating rigid body. K' and K are coordinate systems that correspond to the laboratory frame of reference and the frame of reference attached to our rigid body. He we assume that K may be rotating but there is no translational motion within K.



In the lab frame the velocity of of the i-th particle is

$$\vec{v}_i' = \dot{\vec{R}} + \vec{\omega} \times \vec{r}_i$$

For an observer in the lab frame the total kinetic energy is

$$T' = \frac{1}{2}\sum_i m_i v_i'^2 = \frac{1}{2}\underbrace{\left(\sum_i m_i\right)}_{M}\dot{\vec{R}}^2 + \dot{\vec{R}}\cdot\left(\vec{\omega}\times\underbrace{\sum_i m_i \vec{r}_i}_{M\cdot\vec{r}_{cm}}\right)$$

$$+ \frac{1}{2}\sum_i m_i (\vec{\omega}\times\vec{r}_i)^2$$

A great simplification results if we choose K to be the center of mass, so that $\vec{r}_{CM} = 0$

With this choice

$$T' = \underbrace{T_{transl.}}_{\substack{\text{motion of the}\\ \text{system as a whole}}} + \underbrace{T_{rot}}_{\substack{\text{rotation about}\\ \text{the C.M.}}} \qquad \text{where} \qquad \begin{aligned} T_{trans} &= \tfrac{1}{2} M \dot{\vec{R}}^2 \\ T_{rot} &= \tfrac{1}{2} \sum_i m_i (\vec{\omega} \times \vec{r}_i)^2 \end{aligned}$$

From now on we will assume that K is located at the center of mass. Let us turn our attention to the rotation in the K frame. Using the vector identity

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{B} \cdot \vec{C})(\vec{A} \cdot \vec{D})$$

we can write

$$(\vec{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i) = \cancel{\vec{\omega} \cdot (\vec{r}_i \times (\vec{\omega} \times \vec{r}_i))} = \vec{\omega} \cdot \left(r_i^2 \vec{\omega} - (\vec{\omega} \cdot \vec{r}_i) \vec{r}_i \right)$$

The rotational part of the kinetic energy is then

$$T_{rot} = \frac{1}{2} {\sum_i}' m_i \left({\sum_{\alpha=1}^{3}}' r_{i\alpha}^2 {\sum_{\beta=1}^{3}}' \omega_\beta \omega_\beta - {\sum_{\alpha=1}^{3}}' r_{i\alpha} \omega_\alpha \sum_{\beta=1}^{3} r_{i\beta} \omega_\beta \right)$$

we can also write it as

$$T_{rot} = \frac{1}{2} \sum_{\alpha,\beta} \omega_\alpha \omega_\beta I_{\alpha\beta} = \frac{1}{2} \vec{\omega}^T I \omega$$

where

$$I_{\alpha\beta} = {\sum_i}' m_i \left(r_i^2 \delta_{\alpha\beta} - r_{i\alpha} r_{i\beta} \right) \quad \leftarrow \text{tensor of inertia}$$

or, explicitly

$$I = \begin{pmatrix} {\sum}' m_i (y_i^2 + z_i^2) & -\sum m_i x_i y_i & -\sum m_i x_i z_i \\ -\sum m_i x_i y_i & \sum m_i (x_i^2 + z_i^2) & -\sum m_i y_i z_i \\ -\sum m_i x_i z_i & -\sum m_i y_i z_i & \sum m_i (x_i^2 + y_i^2) \end{pmatrix}$$

The concept of the tensor of inertia can be generalized to the case of continuous mass distribution. For a point mass dm

$$dI = dm \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} \qquad dm = \rho(\vec{r})\, dx\, dy\, dz$$

Then, for example,

$$I_{12} = -\int_V xy\, \rho(\vec{r})\, dx\, dy\, dz \qquad I_{33} = \int_V (x^2+y^2)\, \rho(\vec{r})\, dx\, dy\, dz$$

Since the tensor of inertia is symmetric, $I_{\alpha\beta} = I_{\beta\alpha}$, there are only six independent components. By choosing a proper orthogonal transformation (i.e. by rotating the K frame by some angle) it is possible to diagonalize I; so that

$$UIU^T = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

I_1, I_2, and I_3 are called the principal moments. They are all positive (non-negative)

Displaced axis theorem is a generalization of the familiar parallel axis theorem. It gives the tensor of inertia about an origin displaced by a constant vector $\vec{a}$:

$$I_{\vec{a}} = I_{cm} + M\left(a^2 \delta_{\alpha\beta} - a_\alpha a_\beta\right)$$

Now let us consider the angular momentum of a rigid body. Again, we will use body coordinates with the center of mass at the origin.

$$\vec{r}_i' = \vec{R} + \vec{r}_i$$

$$\vec{v}_i' = \vec{V}_{cm} + \vec{\omega} \times \vec{r}_i \qquad \vec{V}_{CM} \equiv \dot{\vec{R}}$$

$$\vec{L}' \equiv \sum_i \vec{r}_i' \times \vec{p}_i' = \sum_i m_i (\vec{r}_i' \times \vec{v}_i') =$$

$$= \vec{R} \times \vec{P} + \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

where $\vec{P} = M\vec{V}_{CM}$. Hence

$$\vec{L}' = \vec{L}_{CM} + \vec{L}_{rot} \qquad \text{with} \qquad \vec{L}_{CM} = \vec{R} \times \vec{P}$$

The second term can be simplified using the identity $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$$\vec{L}_{rot} = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \sum_i m_i (r_i^2 \vec{\omega} - (\vec{\omega} \cdot \vec{r}_i) \vec{r}_i) =$$

$$= \sum_i m_i \left(\Big[\sum_\alpha r_{i\alpha}^2\Big] \sum_\beta \omega_\beta \vec{e}_\beta - \Big[\sum_\alpha \omega_\alpha r_{i\alpha}\Big] \sum_\beta r_{i\beta} \vec{e}_\beta \right)$$

or

$$\vec{L}_{rot,\beta} = \sum_\alpha I_{\alpha\beta} \omega_\beta \qquad \text{or} \qquad \vec{L}_{rot} = I \vec{\omega}$$

It should be noted that in general $\vec{L}_{rot}$ and $\vec{\omega}$ are not aligned with each other