The Eulerian angles

In order to describe the orientation of a rigid body in space relative to an observer outside the body (lab frame) we must specify how the orientation of the body relative to a fixed coordinate system changes in time. While there are various ways to do so, a commonly chosen scheme uses three angles, ϕ, θ, and ψ, to relate the direction of the principal axes of the body relative to the fixed (lab) frame.

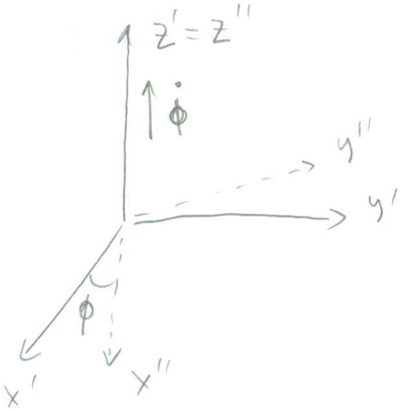
The transformation from one coordinate system to another can be represented in a matrix form:

$$\vec{r} = U\vec{r}\,'$$

Here $\vec{r}\,'$ denotes the position in the fixed system and $\vec{r}$ denotes the position in the body system. The rotation matrix, U, contains three independent angles.

The Eulerian angles are generated in the following series of rotations which take fixed system into the body system.

1. The first rotation is counterclockwise through an angle ϕ about the z-axis. Because the rotation takes place in the xy-plane the transformation matrix is

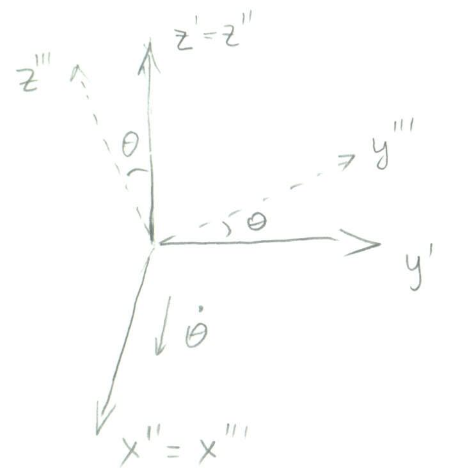


$$U_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\vec{r}\,'' = U_\phi \vec{r}\,'$$

2. The second rotation is counterclockwise through an angle θ about (the new) x'' axis. The transformation matrix is

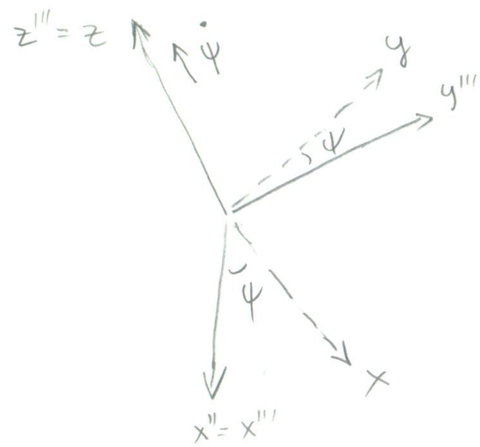


$$U_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

and

$$\vec{r}\,''' = U_\theta \vec{r}\,''$$

3. The third rotation is counterclockwise trough an angle ψ about the z''' axis. The transformation axis is



$$U_\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\vec{r} = U_\psi \vec{r}\,'''$$

The complete transformation is given by

$$\vec{r} = U_\psi U_\theta U_\phi \vec{r}\,' = U \vec{r}\,'$$

The components of this matrix are

$U_{11} = \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi$

$U_{21} = -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi$

$U_{31} = \sin\theta\sin\phi$

$U_{12} = \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi$

$U_{22} = -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi$

$U_{32} = -\sin\theta\cos\phi$

$U_{13} = \sin\psi\sin\theta$

$U_{23} = \cos\psi\sin\theta$

$U_{33} = \cos\theta$

We can also find the relation between the time derivatives of the rotation angles

$$\omega_\phi \equiv \dot{\phi} \qquad \omega_\theta \equiv \dot{\theta} \qquad \omega_\psi \equiv \dot{\psi}$$

and the components of the angular velocity, $\vec{\omega}$, in the body coordinate system. Note that the angular velocities $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ are directed along the following angles

$\dot{\phi}$ – along z' (fixed frame) axis

$\dot{\theta}$ – along the line of nodes (intersection xy and $x'y'$ planes)

$\dot{\psi}$ – along the z (body frame) axis

The components of these angular velocities in the body coordinates are

$$\begin{cases} \dot{\phi}_1 = \dot{\phi} \sin\theta \sin\psi \\ \dot{\phi}_2 = \dot{\phi} \sin\theta \cos\psi \\ \dot{\phi}_3 = \dot{\phi} \cos\theta \end{cases} \qquad \begin{cases} \dot{\theta}_1 = \dot{\theta} \cos\psi \\ \dot{\theta}_2 = -\dot{\theta} \sin\psi \\ \dot{\theta}_3 = 0 \end{cases} \qquad \begin{cases} \dot{\psi}_1 = 0 \\ \dot{\psi}_2 = 0 \\ \dot{\psi}_3 = \psi \end{cases}$$

Collecting the individual components we have

$$\omega_1 = \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$$

$$\omega_2 = \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi$$

$$\omega_3 = \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi} \cos\theta + \dot{\psi}$$