The Euler equations for a rigid body

Let us consider the torque-free motion of a rigid body. In such a case the potential energy V vanishes and the Lagrangian L becomes identical to the rotational kinetic energy. If we choose the coordinate system that correspond to the principal axes of rotation we have

$$T = \frac{1}{2} \sum_i{}' I_i \omega_i^2 \qquad (*)$$

If we choose the Eulerian angles as the generalized coordinates, then the Lagrange's equation for ψ is

$$\frac{\partial T}{\partial \psi} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) = 0$$

It can also be expressed as

$$\sum_i{}' \frac{\partial T}{\partial \omega_i} \frac{\partial \omega_i}{\partial \psi} - \frac{d}{dt}\left(\sum_i \frac{\partial T}{\partial \omega_i} \frac{\partial \omega_i}{\partial \dot{\psi}}\right) = 0 \qquad (**)$$

If we differentiate $\vec{\omega}$ with respect to ψ and $\dot{\psi}$ we get

$$\begin{cases} \dfrac{\partial \omega_1}{\partial \psi} = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi = \omega_2 \\[2ex] \dfrac{\partial \omega_2}{\partial \psi} = -\dot{\phi} \sin\theta \sin\psi - \dot{\theta} \cos\psi = -\omega_1 \\[2ex] \dfrac{\partial \omega_3}{\partial \psi} = 0 \end{cases}$$

and $\dfrac{\partial \omega_1}{\partial \dot{\psi}} = \dfrac{\partial \omega_2}{\partial \dot{\psi}} = 0 \qquad \dfrac{\partial \omega_3}{\partial \dot{\psi}} = 1$

From $(*)$ we also have $\dfrac{\partial T}{\partial \omega_i} = I_i \omega_i$

Then equation $(**)$ becomes

$$I_1\omega_1\omega_2 + I_2\omega_2(-\omega_1) - \frac{d}{dt} I_3\omega_3 = 0$$

or

$$(I_1 - I_2)\,\omega_1\omega_2 - I_3\dot{\omega}_3 = 0$$

By permuting indicies 1,2,3 we can get relations for $\dot{\omega}_1$ and $\dot{\omega}_2$:

$$\left.\begin{aligned} (I_2 - I_3)\,\omega_2\omega_3 - I_1\dot{\omega}_1 &= 0 \\ (I_3 - I_1)\,\omega_3\omega_1 - I_2\dot{\omega}_2 &= 0 \\ (I_1 - I_2)\,\omega_1\omega_2 - I_3\dot{\omega}_3 &= 0 \end{aligned}\right\} \text{Euler's equation for a torque-free motion}$$

To obtain the Euler equations for the case when torques are present we start with

$$\left(\frac{d\vec{L}'}{dt}\right) = \vec{N} \qquad \text{(prime stands for "fixed" frame)}$$

In a previous lecture we also showed that

$$\left(\frac{d\vec{L}'}{dt}\right) = \left(\frac{d\vec{L}}{dt}\right) + \vec{\omega}\times\vec{L}$$

no prime stands for "body" frame

Hence

$$\frac{d\vec{L}}{dt} + \vec{\omega}\times\vec{L} = \vec{N}$$

If we project this equation on the z-axis we get

$$\dot{L}_z + \omega_x L_y - \omega_y L_x = N_z \qquad (***)$$

However, since we have chosen the coordinate system in such a way that its axes coincide with the principal axes of the body we also have

$$L_i = I_3 \omega_i$$

Then equation (***) becomes

$$I \dot{\omega}_3 - (I_1 - I_2)\omega_1 \omega_2 = N_3$$

We can either repeat this procedure manually and project the $\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = \vec{N}$ equation on y and x axes, or make thing more general by recalling that

$$\vec{a} \times \vec{b} = \varepsilon_{ijk} \vec{e}_i a_j b_k$$

where ε_{ijk} is the Levi-Civita symbol and summation over repeated indicies is assumed

so that

$$\dot{L}_i + \varepsilon_{ijk} \omega_j L_k = N_i \qquad i = 1, 2, 3$$

With this the equations of motions are

$$I_i \frac{d\omega_i}{dt} + \varepsilon_{ijk} \omega_j \omega_k I_k = N_i \qquad i = 1, 2, 3$$

Or, we can write them in the expanded form

$$I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = N_1$$

$$I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) = N_2$$

$$I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = N_3$$