Stability of free rigid body rotation

Let us consider a rigid body with principal moments of inertia such that $I_3 > I_2 > I_1$. We let the body axes to coincide with the principle axes.

Now let us assume that the rotation takes place mostly around the X-axis. In this case we can write

$$\vec{\omega} = \omega_1 \hat{e}_1 + \lambda \hat{e}_2 + \mu \hat{e}_3$$

where λ and μ are small compared with ω_1.

In the abscence of torques the Euler equations become

$$\begin{cases} (I_2 - I_3)\lambda\mu - I_1 \dot{\omega}_1 = 0 \\ (I_3 - I_1)\mu\omega_1 - I_2 \dot{\lambda} = 0 \\ (I_1 - I_2)\lambda\omega_1 - I_3 \dot{\mu} = 0 \end{cases}$$

$\mu\lambda$ is of the second order of smallness so we can neglect such a term. Then from the first eq. we get

$$\dot{\omega}_1 = 0 \quad \Rightarrow \quad \omega_1 = \text{const}$$

The other two equations yield

$$\dot{\lambda} = \left(\frac{I_3 - I_1}{I_2}\omega_1\right)\mu$$

$$\dot{\mu} = \underbrace{\left(\frac{I_1 - I_2}{I_3}\omega_1\right)}_{\text{const}}\lambda$$

The two coupled equations can be solved by differentiating one of them and substituting the first order derivative from another one:

$$\ddot{\lambda} = \left(\frac{I_3 - I_1}{I_2}\omega_1\right)\dot{\mu}$$

$$\ddot{\lambda} + \left(\frac{(I_1-I_3)(I_1-I_2)}{I_2 I_3}\omega_1^2\right)\lambda = 0$$

The solution is then

$$\lambda(t) = A e^{i\Omega_{1\lambda} t} + B e^{-i\Omega_{1\lambda} t} \quad \text{with} \quad \Omega_{1\lambda} = \omega_1\sqrt{\frac{(I_1-I_3)(I_1-I_2)}{I_2 I_3}}$$

Given our assumption, namely $I_3 > I_2 > I_1$, $\Omega_{1\lambda}$ happens to be real. Therefore $\lambda(t)$ is oscillatory and if it was small initially it will not grow beyond certain (small) magnitude with time. We can similarly investigate μ and obtain essentially the same result for it, $\Omega_{1\mu} = \Omega_{1\lambda} = \Omega_1$. Hence we can conclude that the small perturbations introduced to $\vec{\omega}$ that is along the x-axis do not increase with time. Consequently, such a rotation is stable.

If we carry out the same kind of analysis for

$$\vec{\omega} = \omega_3 \hat{e}_3 + \lambda \hat{e}_1 + \mu \hat{e}_2$$

i.e. for the case when the unperturbed $\vec{\omega}$ is along the z-axis, we will arrive at the same conclusion (Ω_3 can be obtained from Ω_1 by permuting $1 \leftrightarrows 2$:

$$\Omega_3 = \omega_3\sqrt{\frac{(I_3-I_2)(I_3-I_1)}{I_1 I_2}}$$

Again, Ω_3 is real and the motion is stable.

However for

$$\vec{\omega} = \omega_2 \hat{e}_2 + \lambda \hat{e}_1 + \mu \hat{e}_3$$

$$\Omega_2 = \omega_2 \sqrt{\frac{(I_2 - I_1)(I_2 - I_3)}{I_1 I_3}} \quad \text{is imaginary}$$

that means $\lambda(t)$ and $\mu(t)$ will not be oscillatory. They will grow exponentially and the motion is unstable.