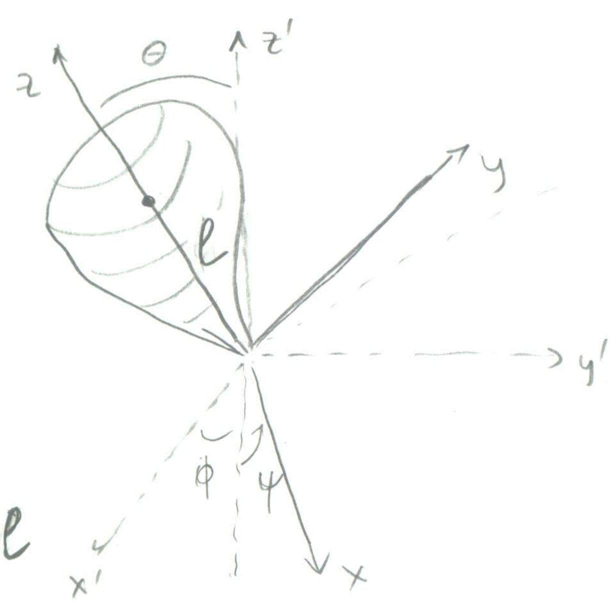
Motion of a symmetric top with one point fixed

Let us consider another example of rigid body rotation — a symmetric top $I_1 = I_2 \neq I_3$ in a uniform gravitational field when one point of the symmetry axis is fixed in space. A wide variety of physical systems, ranging from a child's top to gyroscopes are approximated by such a model.

The configuration of the body is specified by three Euler's angles. The distance of the center of mass (located on the symmetry axis) from the fixed point is denoted by ℓ



θ - inclination of the z axis from the vertical
ϕ - azimuth of the top about the vertical
ψ - rotation of the top about its own z-axis

Let us find the Lagrangian of the system in terms of the Euler's angles. The kinetic energy is

$$T = \frac{1}{2}\sum_i I_i \omega_i^2 = \frac{1}{2} I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2$$

According to our previous consideration of the Euler's angles, the individual components ω_i in the body frame are

$$\omega_1 = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$

$$\omega_2 = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi$$

$$\omega_3 = \dot\phi \cos\theta + \dot\psi$$

Then

$$\omega_1^2 = \dot\phi^2 \sin^2\theta \sin^2\psi + 2\dot\phi\dot\theta \sin\theta \sin\psi \cos\psi + \dot\theta^2 \cos^2\psi$$

$$\omega_2^2 = \dot\phi^2 \sin^2\theta \cos^2\psi - 2\dot\phi\dot\theta \sin\theta \sin\psi \cos\psi + \dot\theta^2 \sin^2\psi$$

so that $\omega_1^2 + \omega_2^2 = \dot\phi^2 \sin^2\theta + \dot\theta^2$

and

$$\omega_3^2 = (\dot\phi \cos\theta + \dot\psi)^2$$

With that

$$T = \frac{1}{2} I_1 (\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} I_3 (\dot\phi \cos\theta + \dot\psi)^2$$

The potential energy is $Mgl\cos\theta$ and the Lagrangian becomes

$$L = \frac{1}{2} I_1 (\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} I_3 (\dot\phi \cos\theta + \dot\psi)^2 - Mgl\cos\theta$$

Both ϕ and ψ are cyclic coordinates. The corresponding conjugated momenta are integrals of motion.

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = (I_1 \sin^2\theta + I_3 \cos^2\theta)\dot\phi + I_3 \dot\psi \cos\theta = \text{const}$$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3 (\dot\psi + \dot\phi \cos\theta) = \text{const}$$

These conjugated momenta are actually the orbital momenta along the $\vec\phi$ and $\vec\psi$ angles. The gravitational torque is directed along the line of nodes and can have no component along either z or z' axis. The equations for p_ϕ and p_ψ can be solved for $\dot\phi$ and $\dot\psi$. For $\dot\psi$ we can write

$$\dot{\psi} = \frac{p_\psi - I_3 \dot{\phi} \cos\theta}{I_3}$$

Substituting this into the equation for p_ϕ gives

$$(I_1 \sin^2\theta + I_3 \cos^2\theta)\dot{\phi} + (p_\psi - I_3 \dot{\phi} \cos\theta)\cos\theta = p_\phi$$

or

$$I_1 \sin^2\theta \, \dot{\phi} + p_\psi \cos\theta = p_\phi$$

so that

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}$$

Similarly $\dot{\psi}$ can be written as

$$\dot{\psi} = \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi \cos\theta)\cos\theta}{I_1 \sin^2\theta}$$

Our system is conservative. Therefore the total energy should be an integral of motion

$$E = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3 \omega_3^2 + Mgl\cos\theta = \text{const}$$

Using the expression for ω_3, i.e. $\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$ we note that the equation for p_ψ can be written as

$$p_\psi = I_3 \omega_3 = \text{const} \qquad \text{or} \qquad I_3 \omega_3^2 = \frac{p_\psi^2}{I_3} = \text{const}$$

Therefore, not only E is an integral of motion but also $E - \frac{1}{2} I_3 \omega_3^2$. We denote it through E'

$$E' \equiv E - \frac{1}{2} I_3 \omega_3^2 = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + Mgl\cos\theta = \text{const}$$

Substituting into this equation the expression for $\dot{\phi}$ above we have

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{(p_\phi - p_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + Mgl\cos\theta$$

The latter can be written as

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta)$$

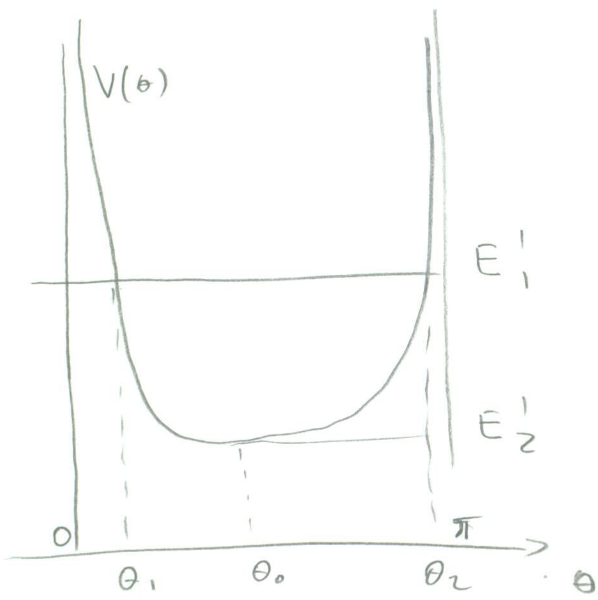
where $V(\theta)$ is an effective potential

$$V(\theta) = \frac{(p_\phi - p_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + Mgl \cos\theta$$

The equation for E' can be solved (at least formally) to yield $t(\theta)$

$$t(\theta) = \int \frac{d\theta}{\sqrt{\frac{2}{I_1}\left[E' - V(\theta)\right]}}$$

and inverted to obtain $\theta(t)$. Then, this $\theta(t)$ can be substituted into the equations for $\dot{\phi}$ and $\dot{\psi}$ to yield $\phi(t)$ and $\psi(t)$. That would constitute a complete solution of the problem. The procedure itself is not very illuminating, however. But we can deduce some qualitative features of the motion by examining the effective potential in a manner analogous to that used for treating the motion of a particle in a central-force field.



θ_1, θ_2 are turning points when $E' = E'_1$

The inclination of the rotating top is confined within $\theta_1 \leq \theta \leq \theta_2$

When $E' = E'_2 = V_{min}$ θ takes only a single value θ_0

In this case the motion is similar to the occurence of circular orbits in the central-force problem θ_0 can be found from

$$\left.\frac{\partial V}{\partial \theta}\right|_{\theta=\theta_0} = 0$$

$$\frac{-\cos\theta_0 (p_\phi - p_\psi \cos\theta_0)^2 + p_\psi \sin^2\theta_0 (p_\phi - p_\psi \cos\theta_0)}{I_1 \sin^3\theta_0} - Mgl\sin\theta_0 = 0$$

If we define $\beta = p_\phi - p_\psi \cos\theta_0$ then it becomes

$$\cos\theta_0\, \beta^2 - p_\psi \sin^2\theta_0\, \beta + Mgl I_1 \sin^4\theta_0 = 0$$

This quadratic equation can be (formally) solved for β:

$$\beta = \frac{p_\psi \sin^2\theta_0}{2\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4MglI_1\cos\theta_0}{p_\psi^2}}\right)$$

Now β must be a real quantity. Therefore we choose the plus sign in the above expression. If $\theta_0 < \frac{\pi}{2}$ then

$$p_\psi^2 \geq 4MglI_1\cos\theta_0$$

However we had it previously that $p_\psi = I_3\omega_3$. Thus,

$$\omega_3 \geq \frac{2}{I_3}\sqrt{MglI_1\cos\theta_0} \qquad \left(\theta_0 < \frac{\pi}{2}\right)$$

We therefore conclude that a steady precession can occur at the fixed angle of inclination θ_0 only if the angular velocity of spin is larger than the limiting value given by the above equation

From the equation for $\dot{\phi}$, i.e. $\dot{\phi} = \dfrac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}$

we can write

$$\dot{\phi}_0 = \frac{\beta}{I_1 \sin^2\theta_0}$$

Hence we have two possible values for the precessional angular velocity $\dot{\phi}_0$, one for each β value.

If ω_3 (or p_ψ) is large then the square root in the expression for β can be expanded into the Taylor series, so that

$$\beta = \frac{p_\psi \sin^2\theta_0}{2\cos\theta_0}\left(1 \pm \left[1 - \frac{1}{2}\,\frac{4 Mgl I_1 \cos\theta_0}{p_\psi^2} + \ldots\right]\right)$$

This gives

$$\dot{\phi}_0^+ = \frac{\beta^+}{I_1 \sin^2\theta_0} = \frac{I_3 \omega_3}{I_1 \cos\theta_0} \quad \leftarrow \text{fast precession}$$

$$\dot{\phi}_0^- = \frac{\beta^-}{I_1 \sin^2\theta_0} = \frac{Mgl}{I_3 \omega_3} \quad \leftarrow \text{slow precession}$$

It is the slower of the two possible precessional angular velocities, $\dot{\phi}_0^-$, that is usually observed