Linear oscillators

A sistem whose Lagrangian has the following form (up to a multiplication constant)

$$L = \frac{\dot{q}^2}{2} - \frac{\omega_0^2 q^2}{2} \qquad \omega_0 = \text{const}$$

is called a simple harmonic oscillator. The equation of motion that results from this Lagrangian is

$$\boxed{\ddot{q} + \omega_0^2 q = 0}$$

or, if we choose a new unit of time $\tau = \omega_0 t$

$$\ddot{q} + q = 0$$

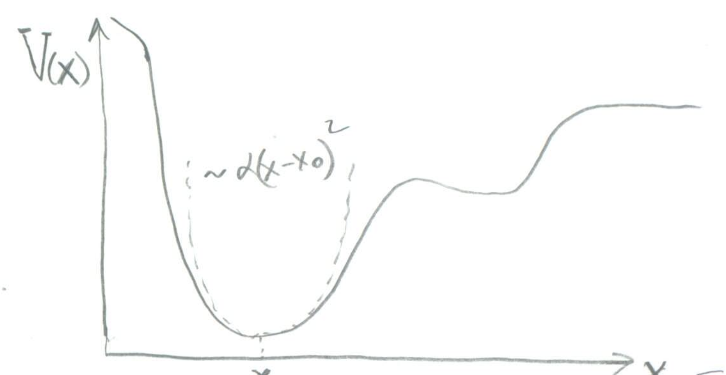
This is a generic equation of motion that discribes many simple models in mechanics. For example, a mass attached to an ideal weighless spring with a spring constant k, has the Lagrangian

$$L = T - V = \frac{m\dot{x}^2}{2} + \frac{kx^2}{2} = \frac{m\dot{x}^2}{2} + \frac{m\omega_0^2}{2}x^2$$

where $\omega_0 = \sqrt{\frac{k}{m}}$

The main reason the harmonic oscillator is so important in physics is the fact that the quadratic form $\sim \alpha x^2$ is the simplest approximation to a smooth potential near its minimum. For an arbitrary potential we can write

$$V(x) \approx V(x_0) + V'(x_0)(x-x_0) + \frac{1}{2!}V''(x_0)(x-x_0)^2 + \frac{1}{3!}V'''(x_0)(x-x_0)^3 + \dots$$



The first term $V(x_0)$ is a constant that serves as a reference of the energy. The equation of motion of a system and its time evolution are independent on the energy reference. We can shift $V(x)$ by any constant arbitrarily without affecting the equation(s) of motion. The second term $V'(x_0)$ vanishes at x_0 becase x_0 is where $V(x)$ has a minimum. Therefore, $\frac{1}{2}V''(x_0)(x-x_0)^2$ is the first nonvanishing and non-constant term in the Taylor expansion of $V(x)$. If we assume that the total mechanical energy lies very close to $V(x_0)$ value, the amplitude of the motion will be small. The motion will be limited to a region where the quadratic approximation matches the original $V(x)$ very closely.

The simple harmonic oscillator equation can be solved easily by a substitution $q(t) = e^{\lambda t}$, which yield the possible values of λ. The solution then can be written as

$$q(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t} = C\sin\omega_0 t + D\cos\omega_0 t =$$

$$= F\sin(\omega_0 t + \phi)$$

where A, B or C, D or F, ϕ are some constants that are determined from the intial condidions of the motion.

Now let us recall a somewhat more complicated system — a damped simple harmonic oscillator

The friction force that causes damping is non-conservative. It either has to be introduced into the equations of motion manually or via a dissipation function $D = \frac{1}{2}\alpha\dot{q}^2$. Then the Lagrange equation reads

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = 0 \qquad \text{where } -\frac{\partial D}{\partial \dot{q}} = F \text{ is the friction force}$$

if $L = \frac{m\dot{q}^2}{2} - \frac{m\omega_0^2 q^2}{2}$ then

$$\ddot{q} + \beta\dot{q} + \omega_0^2 q = 0 \qquad \text{with } \beta \equiv \frac{\alpha}{m}$$

Solving this second order ODE can also be done by a substitution $q = e^{\lambda t}$

$$\lambda^2 + \lambda\beta + \omega_0^2 = 0$$

$$\lambda = \frac{-\beta \pm \sqrt{\beta^2 - 4\omega_0^2}}{2} = -\frac{\beta}{2}\left(1 \pm \sqrt{1 - 4\left(\frac{\omega_0}{\beta}\right)^2}\right)$$

Quantity $Q = \frac{\omega_0}{\beta} = 2\pi \frac{\text{Energy stored}}{\text{Energy lost per cycle}}$ is called the quality factor (Q-factor) of the damped oscillator.

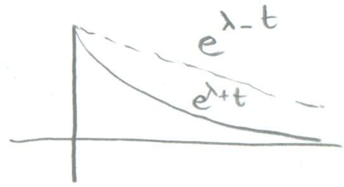
There are three possibilities for the motion depending on the Q value:

1. Overdamped oscillator, $Q < \frac{1}{2}$

The system returns (exponentially decays) to steady state without oscillating. The smaller Q is the slower the system returns to equilibrium

$$q(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t}$$

$$\lambda_\pm < 0$$



2. Underdamped oscillator, $Q > \frac{1}{2}$

The system oscillates (with a slightly different frequency than in the undamped case) with the amplitude gradually decreasing to zero

$$q(t) = A e^{-\frac{\beta}{2}t} \sin(\omega' t + \phi)$$

where $\omega' = \sqrt{\omega_0^2 - \frac{\beta^2}{4}}$

3. Critically damped oscillator, $Q = \frac{1}{2}$.

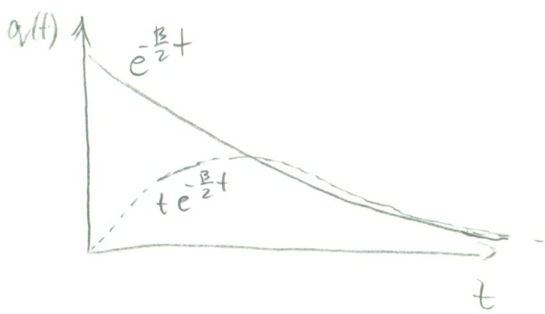
The system returns to equilibrium as quickly as possible without oscillating. This is often desired for the damping of systems such as doors.

$$q(t) = Ate^{-\frac{\beta}{2}t} + Be^{-\frac{\beta}{2}t}$$

The above solution can be obtained as a limiting case of underdamped oscillator when $\omega' \to 0$

$$q(t) = Ce^{-\frac{\beta}{2}t} \sin(\omega' t) + Be^{-\frac{\beta}{2}t} \cos(\omega' t) \approx$$

$$\approx Ce^{-\frac{\beta}{2}t} \omega' t + Be^{-\frac{\beta}{2}t} \cdot 1 = Ate^{-\frac{\beta}{2}t} + Be^{-\frac{\beta}{2}t}$$



What is the physical meaning of Q? Consider case 2) above when $Q \gg 1$. The amplitude is proportional to $e^{-\frac{\beta}{2}t} = e^{-\frac{\omega_0}{2Q}t}$. Since the total energy is proportional to the square of the amplitude we can conclude that it takes $\frac{Q}{2\pi}$ complete oscillations for the energy to decay to $\frac{1}{e}$ of its original value

$$\frac{E_f}{E_i} = \frac{1}{e} = \left(\frac{A\, e^{-\frac{\omega_0}{2Q} t}}{A} \right)^2 = e^{-\frac{\omega_0}{Q} t} \qquad t = \frac{Q}{\omega_0}$$

$$n_{osc} = \frac{t}{T} = \frac{Q}{\omega_0 \frac{2\pi}{\omega_0}} = \frac{Q}{2\pi}$$

The fraction of the stored mechanical energy lost per cycle is

$$\frac{\Delta E}{E} = \frac{E_f - E_i}{E_i} = \frac{\left(e^{-\frac{\omega_0}{2Q} T}\right)^2 - 1}{1} \approx -\frac{2\pi}{Q}$$

In the end of this lecture let us consider an oscillator driven by an external force.

$$\ddot{q} + \omega_0^2 q = \frac{1}{m} F(t)$$

This type of inhomogenious linear differential equation arises very often in physics. The general solution can be found as

$$q_{gen.\,nonhom}(t) = q_{gen\,hom}(t) + q_{partial\,nonhom}(t)$$

$q_{partial\,nonhom.}(t)$ can often be guessed or is known.

As a simple illustrative example let us consider the pendulum (or spring) at rest when a horizontal force F_0 is suddenly turned on

$$F(t) = \begin{cases} 0 & , t < 0 \\ F_0 & , t \geq 0 \end{cases}$$

It is easy to check that

$$q_{\text{part. honhom}}(t) = \frac{F_0}{m\omega_0} \qquad t \geq 0$$

and

$$q_{\text{gen. honhom}}(t) = A \sin(\omega_0 t) + B \cos(\omega_0 t)$$

if $q(t=0) = 0$ and $q'(t=0) = 0$ then we can determine A and B.

$$q(t) = \frac{F_0}{m\omega_0^2} + A \sin \omega_0 t + B \cos \omega_0 t$$

$$q'(t) = \omega_0 A \cos \omega_0 t$$

$$B = -\frac{F_0}{m\omega_0^2} \qquad A = 0$$

$$q(t) = \frac{F_0}{m\omega_0^2}\left(1 - \cos \omega_0 t\right)$$