The general case of n coupled harmonic oscillators

Let us now turn our attention to the general case when we have n degrees of freedom. For simplicity let us denote $\vec{q} = (q_1, \dots q_n)^T$ - set of n generalized coordinates we deal with. Let us assume that our system is conservative and its potential energy is

$$V(q_1, \dots q_n) = V(\vec{q})$$

Given the relation between the Cartesian coordinates and the generalized coordinates

$$\vec{r}_i = \vec{r}_i(q_1, \dots q_n) \qquad \dot{\vec{r}}_i = \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j$$

$$\dot{\vec{r}}_i^{\,2} = \sum_{j=1}^{n} \left(\frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j \right) \sum_{k=1}^{n} \left(\frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k \right)$$

the kinetic energy takes the form

$$T = \frac{1}{2} \sum_i m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2} \sum_{j,k=1}^{n} A_{jk}(\vec{q})\, \dot{q}_j \dot{q}_k$$

where

$$A_{jk} = A_{jk}(q_1 \dots q_n) = \sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$

Our final assumption is that our system is making small oscillations about its equilibrium configuration. If necessary we can always redefine coordinates such that the equilibrium is at $\vec{q} = 0$. The potential energy then can be Taylor expanded around $\vec{q} = 0$

$$V(\vec{q}) = V(0) + \sum_{j=1}^{n} \left.\frac{\partial V}{\partial q_j}\right|_{\vec{q}=0} q_j + \frac{1}{2} \sum_{j,k=1}^{n} \left.\frac{\partial^2 V}{\partial q_j \partial q_k}\right|_{\vec{q}=0} q_j q_k + \dots$$

$V(0)$ can be easily dropped as it is just a constant

All $\left.\frac{\partial V}{\partial q_j}\right|_{\vec{q}=0}$ terms vanish. We are left with

$$V = V(\vec{q}) \approx \frac{1}{2}\sum_{j,k=1}^{n} K_{jk}\, q_j q_k \qquad \leftarrow \text{ quadratic form}$$

Since we want to keep only terms up to the first non-vanishing order we can simplify our kinetic energy. We can ignore everything but but the constant terms in the Taylor expansion of $A_{jk}(\vec{q})$:

$$A_{jk}(\vec{q}) = M_{jk} + \underbrace{\sum_{l=1}^{n} \frac{\partial A_{jk}}{\partial q_l} q_l + \dots}_{\text{ignore}}$$

Then

$$T = T(\dot{\vec{q}}) = \frac{1}{2}\sum_{j,k=1}^{n} M_{jk}\, \dot{q}_j \dot{q}_k$$

and the Lagrangian is

$$L(\vec{q}, \dot{\vec{q}}) = T(\dot{\vec{q}}) - V(\vec{q})$$

Writing the equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, \dots, n$$

gives

$$\frac{\partial L}{\partial q_i} = \sum_{j=1}^{n} K_{ij}\, q_j \qquad i = 1, \dots, n$$

$$\frac{\partial L}{\partial \dot{q}_i} = \sum_{j=1}^{n} M_{ij}\, \dot{q}_j \qquad i = 1, \dots, n$$

$$\sum_{j=1}^{n} M_{ij}\, \ddot{q}_j + \sum_{j} K_{ij}\, q_j \qquad i = 1, \dots, n$$

In the matrix form it looks as

$$M\ddot{\vec{q}} + K\vec{q} = 0 \qquad \vec{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$$

where M and K are $n \times n$ matrices.

We can seek the solution of the above equation in the form

$$\vec{q} = \mathrm{Re}\left[\vec{a}\, e^{i\omega t}\right] \quad \text{where} \quad \vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_1 \end{pmatrix} \text{ is a constant vector-column}$$

That leads to the eigenvalue equation:

$$\left(K - \omega^2 M\right)\vec{a} = 0$$

which has nontrivial solutions only if ω satisfies the secular equation

$$\det\left[K - \omega^2 M\right] = 0$$

The determinant gives an n-th order polynomial in ω^2. Hence we get n solutions, which are the normal frequencies of the system. The corresponding eigenvectors define the normal modes