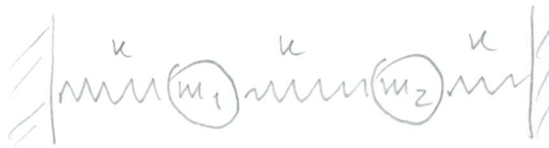
Normal coordinates

Let us now introduce the so called normal coordinates. We will first do it using the example of two equal masses connected by three equal springs — we considered this system in a previous lecture. Let us write again the equations of motion

$$\begin{cases} m\ddot{x}_1 = -2kx_1 + kx_2 \\ m\ddot{x}_2 = kx_1 - 2kx_2 \end{cases}$$



We can see that if we add the two equations or subtract them we get

$$\begin{cases} m(\ddot{x}_1 + \ddot{x}_2) = -k(x_1 + x_2) \\ m(\ddot{x}_1 - \ddot{x}_2) = -3k(x_1 - x_2) \end{cases}$$

or, if we introduce $\xi_1 = \frac{x_1 + x_2}{\sqrt{2}}$ $\xi_2 = \frac{x_1 - x_2}{\sqrt{2}}$ then

$$\begin{cases} m\ddot{\xi}_1 = -k\xi_1 \\ m\ddot{\xi}_2 = -3k\xi_2 \end{cases}$$

The latter two equations are uncoupled. They show that each normal coordinate ξ_i oscillates with a single frequency ω_i $\left(\omega_1 = \sqrt{\frac{k}{m}}, \omega_2 = \sqrt{\frac{3k}{m}}\right)$. In other words the normal coordinates behave just like the coordinates of two independent (uncoupled) oscillators. So going over to the normal coordinates is equivalent to uncoupling oscillators

In the case of x_1 and x_2, as we learned, the equations of motion can be written in the matrix form:

$$M \ddot{\vec{x}} = -\mathbb{K} \vec{x}$$

with

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad \mathbb{K} = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}$$

In the same way the equations for ξ_1 and ξ_2 can be written in the matrix form:

$$M' \ddot{\vec{\xi}} = -\mathbb{K}' \vec{\xi} \qquad \vec{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

The important difference is that both M' and $\mathbb{K}'$ are diagonal

$$M' = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad \mathbb{K}' = \begin{pmatrix} k & 0 \\ 0 & 3k \end{pmatrix}$$

The transition from x_1, x_2 to ξ_1, ξ_2 is said to diagonalize matrices M and $\mathbb{K}$. The fact that the new matrices are diagonal is equivalent to the statement that the equations for ξ_1 and ξ_2 are uncoupled and ξ_1 and ξ_2 oscillate independently.

We can define the normal coordinates more generally in terms of the eigenvectors $\vec{a}$ that describe the motion of the normal modes

$$\vec{a}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \alpha_1 \qquad \vec{a}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \alpha_2$$

where α_i are arbitrary multipliers that define the amplitudes of the normal vibrations. Let us use the normalized eigenvectors though

$$\vec{a}^{(1)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \vec{a}^{(2)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

We can represent $\vec{x}$ as

$$\vec{x} = \xi_1 \vec{a}^{(1)} + \xi_2 \vec{a}^{(2)} = \begin{pmatrix} \frac{\xi_1 + \xi_2}{\sqrt{2}} \\ \frac{\xi_1 - \xi_2}{\sqrt{2}} \end{pmatrix}$$

or we can write it as

$$\vec{x} = A \vec{\xi} \qquad \text{where} \qquad A = \left(\left(\vec{a}^{(1)}\right) \left(\vec{a}^{(2)}\right) \right) - 2\times 2 \text{ matrix}$$

composed of column-vectrons $\vec{a}^{(i)}$

Now we can generalize the concept of normal coordinates for an arbitrary oscillating system with n generalized coordinates $q_1, \dots, q_n$. We know that such a system has n normal modes. Each mode oscillates sinusoidally

$$\vec{q}(t) = \mathrm{Re}\left[\vec{a}^{(i)} e^{i\omega t} \right]$$

where the constant in time column satisfies the eigenvalue equation

$$K \vec{a}^{(i)} = \omega_i^2 M \vec{a}^{(i)}$$

The vector columns $\vec{a}^{(i)}$ are independent and form a basis in n-dimensional space. Thus, any solution of the equations of motion $\vec{q}(t)$ can be expanded in terms of this complete basis

$$\vec{q}(t) = \sum_{i=1}^{n} \xi_i(t) \vec{a}^{(i)} \qquad \text{where} \quad \xi_i = \alpha_i e^{i\omega_i t}$$

Vector-column $\vec{q}(t)$ satisfies the equation of motion in the matrix form

$$M\ddot{\vec{q}} = -\mathbb{K}\vec{q}$$

If we replace $\vec{q}$ with $\vec{q} = \sum_{i=1}^{n} \xi_i \vec{a}^{(i)}$ in the above equation we will get

$$\sum_{i=1}^{n} \ddot{\xi}_i(t)\, M\vec{a}^{(i)} = -\sum_{i=1}^{n} \xi_i(t)\, \mathbb{K}\, \vec{a}^{(i)}$$

or

$$\sum_{i=1}^{n} \xi_i(t)\, \omega_i^2\, M\vec{a}^{(i)} = \sum \xi_i(t)\, \mathbb{K}\, \vec{a}^{(i)}$$

The n vector-columns $\vec{a}^{(i)}$ are independent, so are vector-columns $M\vec{a}^{(i)}$ (matrix M has non-zero determinant). Therefore, the above equations can hold for any t only if all coefficients on each side are equal:

$$\ddot{\xi}_i(t) = -\omega_i^2\, \xi_i(t) \quad \leftarrow \text{ n independent oscillators}$$

To summarize, the transformation from the original coordinates $q_1 \dots q_n$ to normal coordinates $\xi_1, \dots, \xi_n$ is the linear transformation that diagonalizes matrix $\mathbb{K}$:

$$\vec{q} = A\vec{\xi} \qquad A = \left(\begin{pmatrix} \\ \vec{a}^{(1)} \\ \\ \end{pmatrix} \dots \begin{pmatrix} \\ \vec{a}^{(n)} \\ \\ \end{pmatrix} \right)$$

$\vec{a}^{(i)}$ are orthonormal, so the inverse transformation is

$$\vec{\xi} = A^{-1}\vec{q} = A^{T}\vec{q}$$

The original equation of motion is

$$M\ddot{\vec{q}} = -K\vec{q}$$

is yielded from the Lagrangian that is a sum of two quadratic forms

$$L = \frac{1}{2}\dot{\vec{q}}^{T} M \dot{\vec{q}} - \frac{1}{2}\vec{q}^{T} K \vec{q}$$

Now if we substitute $\vec{q} = A\vec{\xi}$ we obtain

$$L = \frac{1}{2}\dot{\vec{\xi}}^{T} \underbrace{A^{T} M A}_{M'} \dot{\vec{\xi}} - \frac{1}{2}\vec{\xi}^{T} \underbrace{A^{T} K A}_{K'} \vec{\xi} = \frac{1}{2}\dot{\vec{\xi}}^{T} M' \dot{\vec{\xi}} - \frac{1}{2}\vec{\xi} K' \vec{\xi}$$

Matrices M' and K' are diagonal

If we write the Lagrange equations using the transformed Lagrangian they will all be uncoupled:

$$M'\ddot{\vec{\xi}} + K'\vec{\xi} = 0$$