Oscillation of n point masses connected by a string (linear array of coupled harmonic oscillators)

Let us consider a model for a solid state system containing many particles bound by interparticle (inter-atomic) potential. When the particles do not get displaced by much from their equilibrium positions the interparticle potential can be approximated well by a quadratic function. Thus, such a system is essentially a collection of coupled oscillators. This model may serve for describing a continuos medium, the propagation of waves through a continuous medium or the vibrations of a crystalline lattice. We will consider a motion of a light elastic string fixed at both ends and loaded with n particles of mass m that are equally spaced along the string.

The displacements of those n particles will be labelled $q_1, \dots q_n$. Two types of displacements are possible: longitudinal and transverse. For simplicity we will assume that the motion is either longitudinal or transverse (although in an actual physical system a combination of the two occur)

The kinetic enegy is given by

$$T = \frac{m}{2}\left(\dot{q}_1^2 + \dots + \dot{q}_n^2\right)$$

In the longitudinal regime the stretch of the section of string between particle j and $j+1$ is $q_{j+1} - q_j$ and $\frac{1}{2}k\left(q_{j+1} - q_j\right)^2$ is the potential energy of that section of the string

In the transverse regime the distance between particles $j+1$ and j is

$$\sqrt{b^2 + (q_{j+1} - q_j)^2} = b\sqrt{1 + \frac{(q_{j+1} - q_j)^2}{b^2}} \approx b + \frac{(q_{j+1} - q_j)^2}{2b} + \dots$$

Hence the stretch is approximately $\frac{(q_{j+1} - q_j)^2}{2b}$ and the potential energy associated with this stretch is also of a quadratic form: $V = F\Delta S = \frac{F}{2b}(q_{j+1} - q_j)^2$

The total potential energy is then

$$V = \frac{k}{2}\left[q_1^2 + (q_2 - q_1)^2 + \dots + (q_n - q_{n-1})^2 + q_n^2\right]$$

where

$$k = \begin{cases} \dfrac{F}{b} = \dfrac{\text{tention in the string}}{\text{separation of particles}} & \leftarrow \text{transverse regime} \\ k = \text{elastis constant} & \leftarrow \text{longitudinal regime} \end{cases}$$

The Lagrangian of our system is

$$L = T - V = \frac{1}{2}\sum_{j=1}^{n}\left[m\dot{q}_j^2 - k(q_{j+1} - q_j)^2\right]$$

With this Lagrangian the equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j}$$

become

$$m\ddot{q} = -k(q_j - q_{j-1}) + k(q_{j+1} - q_j) \qquad j = 1, \dots, n$$

To solve this system of equations we use the ansatz $q_j = \mathrm{Re}\left[a_j\, e^{i\omega t}\right]$ where a_j is the amplitude of vibration for j-th particle. As a result of substitution we get the following recursion formula

$$-m\omega^2 a_j = k\left(a_{j-1} - 2a_j + a_{j+1}\right) \qquad (*)$$

At the endpoints we set $a_0 = a_{n+1} = 0$

The secular equation for ω^2 is then

$$\begin{vmatrix} 2k - m\omega^2 & -k & 0 & \cdots & 0 \\ -k & 2k - m\omega^2 & -k & \cdots & 0 \\ 0 & -k & 2k - m\omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 2k - m\omega^2 \end{vmatrix} = 0$$

We can find the roots of the secular equation working with the simpler determinant:

$$D_n = \begin{vmatrix} \alpha & -1 & 0 & \cdots & 0 \\ -1 & \alpha & -1 & \cdots & 0 \\ 0 & -1 & \alpha & \cdots & 0 \\ \vdots & & & & \alpha \\ 0 & 0 & 0 & \cdots & \end{vmatrix} = 0 \qquad \text{where } \alpha = 2 - \frac{m\omega^2}{k}$$

Expanding the determinant of order n with respect to the first row we get

$$D_n = \alpha \underbrace{\left.\begin{vmatrix} \alpha & -1 & 0 & \cdots & 0 \\ -1 & \alpha & -1 & \cdots & 0 \\ 0 & -1 & \alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \alpha \end{vmatrix}\right\} n-1}_{n-1} - (-1)\cdot \underbrace{\left.\begin{vmatrix} -1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha & -1 & 0 & \cdots & 0 \\ 0 & -1 & \alpha & -1 & \cdots & 0 \\ 0 & 0 & -1 & \alpha & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & \alpha \end{vmatrix}\right\} n-1}_{n-1}$$

The first determinant is D_{n-1} while the second one is $(-1)\cdot D_{n-2}$, i.e.

$$D_n = \alpha D_{n-1} - D_{n-2} \qquad n \geq 2 \qquad (**)$$

Now it is easy to see that $D_1 = \alpha$ and

$$D_2 = \begin{vmatrix} \alpha & -1 \\ -1 & \alpha \end{vmatrix} = \alpha^2 - 1.$$

and we can formally set $D_0 = 1$.

If we make a substitution in equation $(**)$ in the form

$$D_n = p^n$$

where p could, in principle be a function of n (although it will turn out to be not), then equation $(**)$ yields

$$p^n = \alpha p^{n-1} - p^{n-2}$$

or

$$p^2 - \alpha p + 1 = 0 \implies p = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}$$

Now substituting $\alpha = 2\cos\delta$ we obtain for p

$$p = \cos\delta \pm \sqrt{\cos^2\delta - 1} = \cos\delta \pm i\sin\delta = e^{\pm i\delta}$$

Then

$$D_n = p^n = e^{\pm in\delta} = \cos n\delta \pm i\sin n\delta$$

Since equation $(**)$ is homogeneous, the general solution is a linear combination

$$D_n = B\cos n\delta + C\sin n\delta$$

Since $D_0 = 1$ and $D_1 = \alpha = 2\cos\delta$, B and C are

$$B = 1 \qquad C = \cot\delta$$

and

$$D_n = \cos n\delta + \frac{\sin n\delta \cos\delta}{\sin\delta}$$

Using the relation

$$\sin\delta\cos n\delta + \sin n\delta\cos\delta = \sin(\delta + n\delta) = \sin(n+1)\delta$$

we obtain

$$D_n = \frac{\sin(n+1)\delta}{\sin\delta}$$

Now going back to the nontrivial solution of the secular equation we must have $D_n = 0$, or

$$\sin(n+1)\delta = 0 \quad \Rightarrow \quad \delta = \delta_s = \frac{s\pi}{n+1} \qquad s = 1, \dots, n$$

($s=0$ drops out since it leads to $\delta_0 = 0$ and hence to $D_n = n+1 \neq 0$)

Then

$$\alpha = 2 - \frac{m\omega^2}{k} = 2\cos\frac{s\pi}{n+1}$$

and ω is calculated from

$$\omega_s^2 = \frac{2k}{m}\left(1 - \cos\frac{s\pi}{n+1}\right) \qquad s = 1, \dots n$$

$$\omega_s = \sqrt{\frac{2k}{m}}\sqrt{1 - \cos\frac{s\pi}{n+1}}$$

These are the normal frequencies of the system

The last expression allows simplification using the trigonometric identity $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$:

$$\omega_s = 2\omega_0 \sin\frac{s\pi}{2(n+1)} \qquad \text{where} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

If we insert the previous expression for ω_s in the recursion formula (*) which reads $-m\omega^2 a_j = k(a_{j-1} - 2a_j + a_{j+1})$ then

$$-2\left(1 - \cos\frac{s\pi}{n+1}\right)a_j = a_{j-1} - 2a_j + a_{j+1}$$

or

$$a_{j-1} - 2a_j \cos\frac{s\pi}{n+1} + a_{j+1} = 0$$

$$2a_1 \cos\frac{s\pi}{n+1} = a_2 \qquad (a_0 = 0)$$

$$2a_n \cos\frac{s\pi}{n+1} = a_{n-1} \qquad (a_{n+1} = 0)$$

The system of equations for a_j is essentially the same as that for determinants D_n, with $\alpha = 2\cos\frac{s\pi}{n+1} = 2\cos\delta_n$, only the boundary conditions are different. The general solution for a_j is then

$$a_j = A'\cos j\delta_n + A\sin j\delta_n$$

$$= A'\cos\frac{s\pi j}{n+1} + A\sin\frac{s\pi j}{n+1}$$

Points $j=0$ and $j=n+1$ are tightly clamped, so that $a_0 = a_{n+1} = 0$. Then for $j=0$ we obtain

$A' = 0$ and $a_j = A\sin\frac{s\pi j}{n+1}$

This yield the following for q_j

$$q_j = A\sin\frac{s\pi j}{n+1}\cos\omega_s t$$

The general type of motion is a linear combination of all the normal modes:

$$q_j = \sum_{s=1}^{n} A_s \sin\left(\frac{s\pi j}{n+1}\right)\cos(\omega_s t - \phi_s)$$

Where constants A_s and ϕ_s are determined from the initial conditions

Now suppose we deal with the case when n is very large. Then for $s \ll n$

$$\omega_s = 2\omega_0 \sin\frac{s\pi}{2(n+1)} \approx \omega_0 \frac{s\pi}{n+1}$$

For transverse oscillations

$$\omega_0 = \sqrt{\frac{K}{m}} = \sqrt{\frac{F}{m b}}$$

Equating these two yields

$$\omega_s \approx \sqrt{\frac{F}{m/b}}\ \underbrace{\frac{s\pi}{(n+1)b}}_{l \leftarrow \text{ total length of the string}} = \sqrt{\frac{F}{\mu}}\ \frac{s\pi}{l} \qquad s = 1, 2, \dots$$

$\mu \leftarrow$ linear density

In particular

$$\omega_1 = \frac{\pi}{l}\sqrt{\frac{F}{\mu}}$$

Now let us look at the displacement of the string. Instead of denoting particles by ~~them~~ j values, let us denote them by their distance down the string from the fixed end $x = jb$

$$\frac{s\pi j}{n+1} = \frac{s\pi\, jb}{(n+1)b} = \frac{s\pi x}{l}$$

Then

$$q_s(x,t) = A_s \sin\left(\frac{s\pi x}{l}\right)\cos\omega_s t \qquad s = 1, 2, \dots$$

$$= A_s \sin\left(\frac{2\pi x}{\lambda_s}\right)\cos 2\pi f_s t \quad \leftarrow \text{standing wave of wavelength } \lambda_s$$

where $\lambda_s = \frac{2l}{s}$ and $f_s = \frac{\omega_s}{2\pi}$ are the "wavelength" and "frequency".