Hamiltonian mechanics (continue)

As yet another simple example of the Hamiltonian formalism let us consider a particle in a central force field.

By conservation of angular momentum the motion is confined to a plane. We can define the polar coordinates r and ϕ in this plane. The kinetic energy is then

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$$

We must now express T in terms of generalized momenta p_r and p_ϕ. According to the definition

$$p_r \equiv \frac{\partial L}{\partial \dot{r}} = \frac{\partial T}{\partial \dot{r}} = m\dot{r} \qquad \text{(we assume } V = V(r)\text{)}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi}$$

p_r is here is just the radial component of $m\vec{v}$ (the ordinary momentum), but p_ϕ is the angular momentum. Now we must solve for $\dot{r}$ and $\dot{\phi}$ in terms of p_r and p_ϕ

$$\dot{r} = \frac{p_r}{m} \qquad \dot{\phi} = \frac{p_\phi}{mr^2}$$

and substitute them into the original expression for T

$$T = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right)$$

which gives the following Hamiltonian

$$H = T + V = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r)$$

With that we can now write down all four Hamilton equations:

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2} \qquad \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

The first of these reproduces the definition of the radial momentum. The second (if we substitute the first into it) gives as the familiar result that $m\ddot{r}$ is the sum of the actuar radial force and the centrifugal term $\frac{p_\phi^2}{mr^3}$. The third equation reproduces the definition of p_ϕ. The forth one tells us that the angular momentum is conserved.

This exomple with a particle in a central field illustrates again the general algorithm for setting up the Hamilton equations:

1. Choose generalized coordinates $q_1, \ldots, q_n$
2. Write down the Lagrangian in terms of q_i and $\dot{q}_i$, $i = 1 \ldots n$.
3. Find the conjugated momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$ $i = 1 \ldots n$
4. Solve for $\dot{q}_i$ in terms of p's and q's, $i = 1 \ldots n$
5. Write down the Hamiltonian as a function of p's and q's, $H = \sum_i p_i \dot{q}_i - L(\vec{q}, \dot{\vec{q}}(\vec{p}, \vec{q}))$, by replacing all $\dot{q}_i$'s with the expressions obtained in step 4
6. Write down Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Looking at the list of steps we must perform, it may not be immedialy clear what advantege the Hamiltonian method has over the Lagrangian method. One of the advantages is revealed when we deal with cyclic (also called ignorable) coordinates, i.e. such coordinates that $\frac{\partial L}{\partial q_i} = 0$

It follows immediately from the equation

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

that the conjugate momenta corresponding to q_i are constants (integrals of motion). This greatly simplifies solving the equations of motion for systems with cyclic coordinates in the Hamiltonian formalism because some of the equations become trivial to solve

The second advantage of the Hamiltonian form of the equatoons of motion comes from the fact that it allows more freedom in choosing the generalized coordinates

Suppose we have a system with n degrees of freedom. The Lagrangian formalism gives n second-order ordinary differential equations for variables $q_1, \dots q_n$. The Hamiltonian formalism yields $2n$ first-order differential equations for $q_1, \dots q_n, p_1, \dots p_n$. There is equivalency here because any set of second-order equations can be recast as twice as many first-order equations. For simplicity let us assume we have only one degree of freedom. The Lagrange equation could be written as

$$f(\ddot{q}, \dot{q}, q) = 0$$

where f is some function. If we define

$$s = \dot{q}$$

then $\dot{s} = \ddot{q}$ and the original equation becomes

$$f(\dot{s}, s, q) = 0$$

Hence the second-order equation $f(\ddot{q}, \dot{q}, q) = 0$ is replaced with two first order equations $\dot{q} = s$ and $f(\dot{s}, s, q) = 0$.

The fact that we reduce a second order equation to two first-order equations by itself does not constitute an advantage. However the specific form of Hamilton's equations is a big improvement. We can combine the equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = f_i(\vec{q}, \vec{p}) \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = g_i(\vec{q}, \vec{p})$$

in one equation written in the vector form

$$\dot{\vec{q}} = \vec{f}(\vec{q}, \vec{p}) \quad \dot{\vec{p}} = \vec{g}(\vec{q}, \vec{p}) \implies \dot{\vec{u}} = \vec{h}(\vec{u})$$

where $\vec{u} = (\vec{q}, \vec{p}) = (q_1, \dots q_n, p_1, \dots p_n)$ is a $2n$-dimensional vector (phase-space vector) that contains all generalized coordinates and the conjugated momenta, while $\vec{h}$ is a vector comprising $2n$ functions $f_1, \dots f_n$ and $g_1, \dots g_n$.

Treating the n position coordinates $\vec{q}$ on an equal footing with n momenta $\vec{p}$ (i.e. forming a single phase space vector $\vec{u}$) gives additional flexibility. We know that any set of generalized coordinates $\vec{q} = (q_1, \dots q_n)$ can be replaced by a second (presumably more convenient) set $Q = (Q_1, \dots Q_n)$ where each Q_i is a function of $q_1, \dots q_n$:

$$\vec{Q} = \vec{Q}(\vec{q})$$

The Lagrange equations will have the same form,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i} = \frac{\partial L}{\partial Q_i}$$

as they had in the original coordinates. In other words the form of Lagrange equations is invariant with respect to the transformations

$$\vec{Q} = \vec{Q}(\vec{q}).$$

The Hamiltonian formalism shares this flexibility - the Hamilton equations are invariant with respect to the same transformations of generalized coordinates. However, the Hamiltonian formalism also allows even more general transformations that mix coordinates and momenta:

$$\vec{Q} = \vec{Q}(\vec{q},\vec{p}) \quad \text{and} \quad \vec{P} = \vec{P}(\vec{q},\vec{p})$$

If the above transformations satisfy certain conditions the form of the Hamilton equations remains unchanged. Such transformations are called canonical transformations.

Let us establish those conditions. We essentially want that the new set $(\vec{Q},\vec{P})$ was such that the form of the Hamilton equations is preserved, i.e.

$$\dot{\vec{P}} = -\frac{\partial H'}{\partial \vec{Q}} \quad \text{and} \quad \dot{\vec{Q}} = \frac{\partial H'}{\partial \vec{P}} \quad \text{where } H'(\vec{P},\vec{Q}) \text{ is}$$

a new Hamiltonian that must be determined. The time derivative of Q_i is

$$\dot{Q}_i = \frac{\partial Q_i}{\partial \vec{q}}\cdot\dot{\vec{q}} + \frac{\partial Q_i}{\partial \vec{p}}\cdot\dot{\vec{p}} = \frac{\partial Q_i}{\partial \vec{q}}\cdot\frac{\partial H}{\partial \vec{p}} - \frac{\partial Q_i}{\partial \vec{p}}\cdot\frac{\partial H}{\partial \vec{q}} = \{Q_i, H\} \text{ where}$$

$\{.,.\}$ is called the Poisson bracket and this construct is related to a commutator in quantum mechanics.

We also have the identity for P_i:

$$\frac{\partial H}{\partial P_i} = \frac{\partial H}{\partial \vec{q}} \cdot \frac{\partial \vec{q}}{\partial P_i} + \frac{\partial H}{\partial \vec{p}} \cdot \frac{\partial \vec{p}}{\partial P_i}$$

If the transformation is canonical the two last expressions must be equal, that is

$$\left(\frac{\partial Q_i}{\partial p_j}\right)_{\vec{q},\vec{p}} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{\vec{Q},\vec{P}}$$

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{\vec{q},\vec{p}} = \left(\frac{\partial p_j}{\partial P_i}\right)_{\vec{Q},\vec{P}}$$

Similar considerations for the generalized momenta P_i lead to two other conditions:

$$\left(\frac{\partial P_i}{\partial p_j}\right)_{\vec{q},\vec{p}} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{\vec{Q},\vec{P}}$$

$$\left(\frac{\partial P_i}{\partial q_j}\right)_{\vec{q},\vec{p}} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{\vec{Q},\vec{P}}$$