Liouville's theorem

In a previous lecture it was mentioned that the n-dimensional configuration space $q_1, \dots q_n$ can be combined with the n-dimensional momentum space $p_1, \dots p_n$ to form 2n-dimensional phase space. A point in this 2n-dimensional space unambiguously specifies the mechanical state of the system under consideration. For example, consider a harmonic oscillator with the Hamiltonian

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

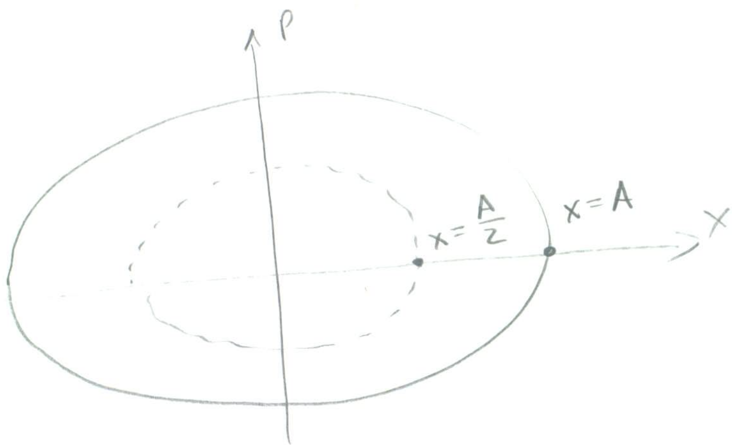
The Hamilton's equation for this system are

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x$$

and their solution is

$$x = A \cos(\omega t + \phi) \qquad p = m\dot{x} = -m\omega A \sin(\omega t + \phi)$$

The orbits in the phase space look as follows

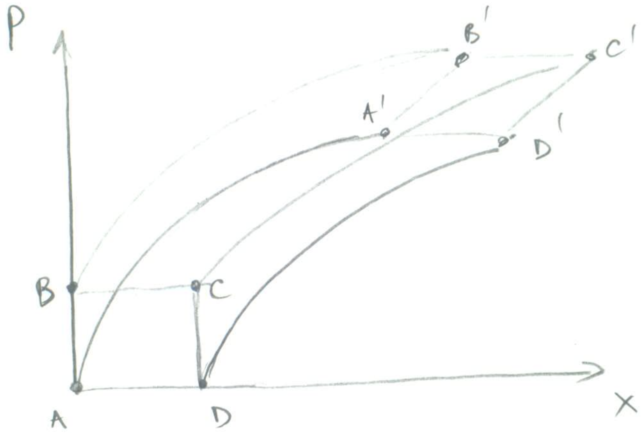


Consider anoter example — a falling body

$$H = T + V = \frac{p^2}{2m} - mgx \qquad \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial H}{\partial x} = mg$$

The solution of Hamilton's equations is

$$p = p_0 + mgt \qquad x = x_0 + \frac{p_0}{m} t + \frac{1}{2} g t^2$$



it turns out that the area of the parallelogram remains constant in time

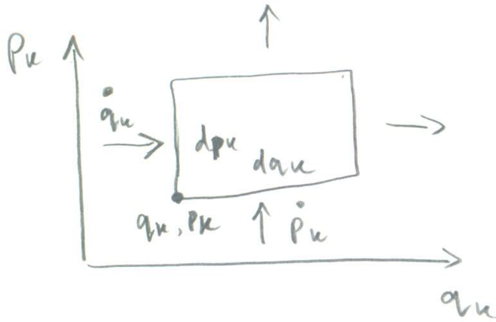
If at a given time the position and momenta of all particles in a system are known then the subsequent motion of the system is completely determined.

For a very large collection of particles (e.g. a system with a very large number of degrees of freedom) it is difficult to identify the particular point in phase space that correctly represents the system. But we may fill the phase space with a collection of points representing possible conditions of the system. Moreover, we can substitute a discussion of an ensemble of equivalent systems.

It is not difficult to realize that the trajectories in the phase space that correspond to two different initial mechanical states (~~two~~ members of the ensemble) cannot intersect. We may consider the representative points to be sufficiently numerous and define their density in phase space, ρ. The number of systems (ensemble members) whose representative points lie within a volume $dq_1 \ldots dq_n dp_1 \ldots dp_n$ of phase space is

$$N = \rho \, dq_1 \ldots dq_n dp_1 \ldots dp_n$$

Consider an element of area (subspace) in the $q_n p_n$ plane. The number of representative points moving across the left-hand edge into the unit area per time is



$$\rho \frac{dq_k}{dt} dp_k = \rho \dot q_k \, dp_k$$

and for the lower edge

$$\rho \frac{dp_k}{dt} dq_k = \rho \dot p_k \, dq_k$$

The total number of points is then

$$\rho \left(\dot q_k \, dp_k + \dot p_k \, dq_k \right) \qquad (*)$$

The number of representative points moving out of the area per unit time is by Taylor series expansion:

$$\left[\rho \dot q_k + \frac{\partial}{\partial q_k}(\rho \dot q_k) \, dq_k \right] dp_k + \left[\rho \dot p_k + \frac{\partial}{\partial p_k}(\rho \dot p_k) \, dp_k \right] dq_k \qquad (**)$$

The total increase in density in $dq_k dp_k$ per unit time is the difference between $(*)$ and $(**)$:

$$\frac{\partial \rho}{\partial t} dq_k dp_k = - \left[\frac{\partial}{\partial q_k}(\rho \dot q_k) + \frac{\partial}{\partial p_k}(\rho \dot p_k) \right] dq_k dp_k$$

If we divide the last expression by $dq_k dp_k$ and sum over all possible k we will find

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{n} \left(\frac{\partial \rho}{\partial q_k} \dot q_k + \rho \frac{\partial \dot q_k}{\partial q_k} + \frac{\partial \rho}{\partial p_k} \dot p_k + \rho \frac{\partial \dot p_k}{\partial p_k} \right) = 0$$

From Hamilton's equations $\dot q_k = \frac{\partial H}{\partial p_k}$ $\quad \dot p_k = - \frac{\partial H}{\partial q_k}$ we also have $\frac{\partial \dot q_k}{\partial q_k} = - \frac{\partial \dot p_k}{\partial p_k}$ (second derivative is assumed continuous)

So the above equation becomes

$$\frac{\partial \rho}{\partial t} + \sum_k \left(\frac{\partial \rho}{\partial q_k} \underbrace{\frac{dq_k}{dt}}_{\frac{\partial H}{\partial p_k}} + \frac{\partial \rho}{\partial p_k} \underbrace{\frac{dp_k}{dt}}_{-\frac{\partial H}{\partial q_k}} \right) = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \{\rho, H\} = 0$$

or equivalently, we can state $\frac{d\rho}{dt} = 0$. This is the famous Liouville theorem, which states that the density of representative points in phase space remain constant.

Poisson brackets

An important construct in Hamiltonian mechanics is the so-called Poisson bracket(s). It governs the time-evolution of a system and also has direct relation to an important quantum-mechanical construct — the commutator.

The Poisson bracket is defined as follows

$$\{g, h\} = \sum_{k=1}^{n} \left(\frac{\partial g}{\partial q_k} \frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k} \frac{\partial h}{\partial q_k} \right)$$

where $g(\vec{q}, \vec{p})$ and $h(\vec{q}, \vec{p})$ are any two continuous functions of generalized coordinates $\vec{q} = (q_1, \dots q_n)$ and the conjugated momenta $\vec{p} = (p_1, \dots p_n)$

The Poisson bracket has several properties that resemble those of commutators

1) $\dfrac{dg}{dt} = \{g, H\} + \dfrac{\partial g}{\partial t}$

Indeed, according to the definition

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \sum_k \left(\frac{\partial g}{\partial q_k} \underbrace{\frac{\partial q_k}{\partial t}}_{\dot{q}_k = \frac{\partial H}{\partial p_k}} + \frac{\partial g}{\partial p_k} \underbrace{\frac{\partial p_k}{\partial t}}_{\dot{p}_k = -\frac{\partial H}{\partial q_k}} \right) = \frac{\partial g}{\partial t} + \{g, H\}$$

2) $\dot{q}_j = \{q_j, H\} \qquad \dot{p}_j = \{p_j, H\}$

Indeed,

$$\{q_j, H\} = \sum_k \left(\underbrace{\frac{\partial q_j}{\partial q_k}}_{\delta_{jk}} \underbrace{\frac{\partial H}{\partial p_k}}_{\dot{q}_k} - \underbrace{\frac{\partial q_j}{\partial p_k}}_{0} \frac{\partial H}{\partial q_k} \right) = q_j$$

Similarly,

$$\{p_j, H\} = \sum_k \left(\underbrace{\frac{\partial p_j}{\partial q_k}}_{=0} \frac{\partial H}{\partial p_k} - \underbrace{\frac{\partial p_j}{\partial p_k}}_{=\delta_{jk}} \underbrace{\frac{\partial H}{\partial q_k}}_{=-\dot{p}_k} \right) = \dot{p}_j$$

3) $\{p_i, p_j\} = 0 \qquad \{q_i, q_j\} = 0$

$$\{q_i, q_j\} = \sum_k \left(\frac{\partial q_i}{\partial q_k} \underbrace{\frac{\partial q_j}{\partial p_k}}_{=0} - \underbrace{\frac{\partial q_i}{\partial p_k}}_{=0} \frac{\partial q_j}{\partial q_k} \right) = 0$$

$$\{p_i, p_j\} = \sum_k \left(\underbrace{\frac{\partial p_i}{\partial q_k}}_{=0} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \underbrace{\frac{\partial p_j}{\partial q_k}}_{=0} \right) = 0$$

4) $\{q_i, p_j\} = \delta_{ij}$

$$\{q_i, p_j\} = \sum_k \left(\underbrace{\frac{\partial q_i}{\partial q_k}}_{=\delta_{ik}} \underbrace{\frac{\partial p_j}{\partial p_k}}_{=\delta_{jk}} - \underbrace{\frac{\partial q_i}{\partial p_k}}_{=0} \underbrace{\frac{\partial p_j}{\partial q_k}}_{=0} \right) = \delta_{ik}$$

If the Poisson bracket of two quantities vanishes the quantities are said to commute.

5) Any quantity that does not depend explicitly on time and commutes with the Hamiltonian is an integral of motion

This property follows directly from 1)