Hamilton - Jacobi equation

The Hamilton - Jacobi approach is another formulation of classical mechanics equivalent to the Newtonian, Lagrangian, and Hamiltonian formulations. It is particularly suitable for connecting the classical and quantum mechanics.

Before we derive the Hamilton - Jacobi equation based on the Hamilton approach let us first take a glance what it is and its mathematical formulation. It is a first-order, non-linear, partial differential equation in the form

$$H + \frac{\partial S}{\partial t} = 0$$

where H is the Hamiltonian function that depends on a set of generalized coordinates $q_1 \dots q_n$ and partial derivatives of S (called action)

$$H = H\left(q_1, \dots, q_n, \frac{\partial S}{\partial q_1}, \dots \frac{\partial S}{\partial q_n}, t\right)$$

The action, S (also called Hamilton's principal function) is a function of $q_1, \dots q_n$ and t.

$$S = S(q_1, \dots q_n, t)$$

Remarkably, it is equal to the classical action

$S = \int L\, dt$ where L is the Lagrangian

Now let us turn to the derivation of the HJ equation. First, consider the second type of the generating function $F_2(\vec{q}, \vec{P}, t)$. It leads to the following

$$\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - H' + \frac{dF}{dt} \qquad F = F_2(\vec{q}, \vec{P}, t) - \sum_i Q_i P_i$$

$$\sum_i p_i \dot{q}_i - H = -\sum_i Q_i \dot{P}_i - H' + \underbrace{\frac{d}{dt} F_2(q, P, t)}_{\sum_i \frac{\partial F_2}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial F_2}{\partial P_i} \dot{P}_i}$$

So

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i} \qquad H' = H + \frac{\partial F_2}{\partial t} \qquad (*)$$

Let us now try to find such a transformation that the canonical momenta and the corresponding generalized coordinates are constant, i.e.,

$$Q_i = \alpha_i \qquad P_i = \beta_i \qquad i = 1 \dots n$$

and these constants α_i, β_i are given by the initial conditions. When we have found such transformations the transformation equations look as follows

$$q_i = q_i(\vec{\alpha}, \vec{\beta}, t) \qquad p_i = p_i(\vec{\alpha}, \vec{\beta}, t)$$

Coordinates P_i, Q_i obey the Hamilton equations

$$0 = \dot{P}_i = -\frac{\partial H'}{\partial Q_i} \qquad 0 = \dot{Q}_i = \frac{\partial H'}{\partial P_i}$$

These conditions would certainly be fulfilled by the function $H' = 0$. In order to perform the coordinate transformation we need a generating function. For historical reason (Jacobi made this choice) we adopt F_2 out of the four possible types and we will call it $S(\vec{q}, \vec{P}, t)$. If $H' = 0$ then we are left with the equation

$$H + \frac{\partial S}{\partial t} = 0 \quad \leftarrow \text{ Hamilton – Jacobi equation}$$

or, more explicitly,

$$\frac{\partial S(q_1, .. q_n, P_1 = \beta_1, \ldots P_n = \beta_n, t)}{\partial t} + H\left(q_1, .. q_n, \frac{\partial S}{\partial q_1} \cdots \frac{\partial S}{\partial q_n}, t\right) = 0$$

The equation is nonlinear because H depends quadratically on the momenta that enter the derivatives $\frac{\partial S}{\partial q_i}$. It is an equation with $n+1$ variables $(q_1, .. q_n, t)$

To get S we would need to integrate the differential equation $n+1$ times (each derivative $\frac{\partial S}{\partial q_i}$ and $\frac{\partial S}{\partial t}$ requires one integration) and we thus obtain $n+1$ integration constants. But because S appears in the HJ equation only as a derivative, S is determined up to a constant, i.e. $S = S' + a$. This means that one of those $n+1$ integration constants must be additive to S, which is not essential. We thus obtain as a solution function

$$S = S(q_1 \ldots q_n, \beta_1 \ldots, \beta_n, t)$$

where β_i's are integration constants. A comparison with $(*)$ yields

$$P_i = \beta_i \qquad Q_i = \frac{\partial S}{\partial P_i} = \frac{\partial S(q_1 \ldots q_n, \beta_1, \ldots \beta_n, t)}{\partial \beta_i} = \alpha_i$$

The original coordinates result from (*) as follows

$$\alpha_i = \frac{\partial S(\vec{q},\vec{\beta},t)}{\partial \beta_i} \quad \text{gives us} \quad q_i = q_i(\vec{\alpha},\vec{\beta},t)$$

Insertion into

$$p_i = \frac{\partial S(\vec{q},\vec{\beta},t)}{\partial q_i} = p_i(\vec{q},\vec{\beta},t) \quad \text{gives} \quad p_i = p_i(\vec{\alpha},\vec{\beta},t)$$

Now $q_i(\vec{\alpha},\vec{\beta},t)$ and $p_i(\vec{\alpha},\vec{\beta},t)$ are known func-~~t~~tions of time and α_i's and β_i's

We can separate off the time dependence is S when H is not an explicit function of time.

$$-\frac{\partial S}{\partial t} = H = E$$

From this it follows that

$$S(\vec{q},\vec{P}=\vec{\beta},t) = S_0(\vec{q},\vec{P}=\vec{\beta}) - Et$$

To explain the meaning of S let us consider its total derivative with respect to time

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q_i}\dot{q}_i + \sum \frac{\partial S}{\partial P_i}\underbrace{\dot{P}_i}_{0}$$

Now because

$$\frac{\partial S}{\partial q_i} = p_i \quad \text{and} \quad \frac{\partial S}{\partial t} = -H$$

it follows that

$$\frac{dS(\vec{q},\vec{P}(\vec{p},\vec{q})=\vec{\beta},t)}{dt} = \sum_i p_i\dot{q}_i - H = L(\vec{q},\vec{p},t)$$

That means that S is given by the integral

$$S = \int L\,dt + a$$

The Hamilton-Jacobi equation for the harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{kq^2}{2}$$

The action $S = F_2$ is then (see lecture)

$$S = S(q, P, t) \qquad p = \frac{\partial S}{\partial q}$$

The Hamiltonian-Jacobi equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{kq^2}{2} = 0$$

Since the Hamiltonian is not time-dependent we can try the following ansatz to solve that HJ equation

$$S = \gamma(t) + \sigma(q)$$

$$\frac{\partial S}{\partial q} = \frac{d\sigma}{dq} \qquad \frac{\partial S}{\partial t} = \frac{d\gamma}{dt}$$

This leads to

$$-\dot{\gamma}(t) = \frac{1}{2m}\left(\frac{d\sigma}{dq}\right)^2 + \frac{kq^2}{2} = \beta$$

where ε is a separation constant. The left-hand side depends only on time and the right-hand side depends only on q, therefore each side must be a constant. With that we can write

$$\dot{\gamma}(t) = -\beta \Rightarrow \gamma(t) = -\beta t$$

For the space-dependent part we have

$$\frac{1}{2m}\left(\frac{d\sigma}{dq}\right)^2 + \frac{kq^2}{2} = \beta \Rightarrow \frac{d\sigma}{dq} = \sqrt{2m\beta - mkq^2}$$

The entire S is then

$$S(q, \beta, t) = \sqrt{mk} \int \sqrt{\frac{2\beta}{k} - q^2}\, dq - \beta t$$

For $Q \equiv \alpha$ we then have

$$\alpha = Q = \frac{\partial S}{\partial \beta} = \sqrt{\frac{m}{k}} \int \frac{1}{\sqrt{\frac{2\beta}{k} - q^2}}\, dq - t$$

The integral can be evaluated $\left(\int \frac{dx}{\sqrt{c^2 - x^2}} = \arcsin\left(\frac{x}{c}\right) \right)$

and we obtain

$$Q + t = \sqrt{\frac{m}{k}} \arcsin\left(\sqrt{\frac{k}{2\beta}}\, q \right)$$

or

$$q = \sqrt{\frac{2\beta}{k}} \sin\left(\sqrt{\frac{k}{m}}\, t + Q \right) = \sqrt{\frac{2\beta}{k}} \sin\left(\omega t + Q \right)$$