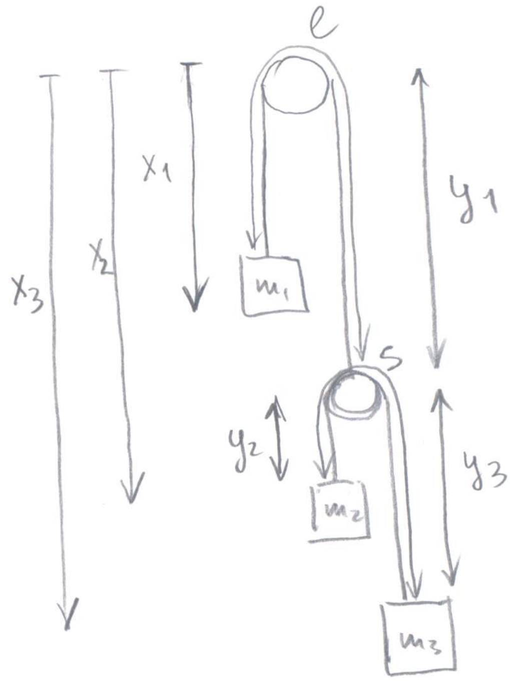
1

$$L = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2} + \frac{m_3 \dot{x}_3^2}{2}$$
$$+ m_1 g x_1 + m_2 g x_2 + m_2 g x_3$$



There are constraints in the system:

$$\begin{cases} x_1 + y_1 = l & \leftarrow \text{length of rope 1} \\ y_2 + y_3 = s & \leftarrow \text{length of rope 2} \\ y_1 + y_2 = x_2 \\ y_1 + y_3 = x_3 \end{cases}$$

Using these constraints we can eliminate one of the x_i's from consideration:

$$\begin{cases} x_1 + y_1 = l \\ y_2 + y_3 = s \\ y_1 + y_2 = x_2 \\ y_1 + y_3 = x_3 \end{cases} \Rightarrow \quad y_1 = l - x_1 \quad \begin{cases} y_2 + y_3 = s \\ l - x_1 + y_2 = x_2 \\ l - x_1 + y_3 = x_3 \end{cases} \Rightarrow \quad y_2 = s - y_3 \quad \begin{cases} l + s - y_3 = x_1 + x_2 \\ l + y_3 = x_1 + x_3 \end{cases} \Rightarrow \quad y_3 = l + s - x_1 - x_2$$

$$x_1 = l + \frac{s}{2} - \frac{x_2}{2} - \frac{x_3}{2}$$

$$\dot{x}_1 = -\frac{1}{2}(\dot{x}_2 + \dot{x}_3)$$

Then our Lagrangian becomes:

$$L = \frac{m_1}{8}(\dot{x}_2 + \dot{x}_3)^2 + \frac{m_2}{2}\dot{x}_2^2 + \frac{m_3}{2}\dot{x}_3^2 + m_1 g \left(l + \frac{s}{2} - \frac{x_2}{2} - \frac{x_3}{2}\right) + m_2 g x_2 + m_3 g x_3$$

$$= \left(\frac{m_2}{2} + \frac{m_1}{8}\right)\dot{x}_2^2 + \left(\frac{m_3}{2} + \frac{m_1}{8}\right)\dot{x}_3^2 + \frac{m_1}{4}\dot{x}_2 \dot{x}_3 + g\left(m_2 - \frac{m_1}{2}\right)x_2 + g\left(m_3 - \frac{m_1}{2}\right)x_3 + m_1 g \left(l + \frac{s}{2}\right)$$

Neither x_2, nor x_3 are cyclic. The partial derivatives are:

$$\frac{\partial L}{\partial x_2} = g\left(m_2 - \frac{m_1}{2}\right) \qquad \frac{\partial L}{\partial x_3} = g\left(m_3 - \frac{m_1}{2}\right)$$

$$\frac{\partial L}{\partial \dot{x}_2} = \left(m_2 + \frac{m_1}{4}\right)\dot{x}_2 + \frac{m_1}{4}\dot{x}_3 \qquad \frac{\partial L}{\partial \dot{x}_3} = \left(m_3 + \frac{m_1}{4}\right)\dot{x}_3 + \frac{m_1}{4}\dot{x}_2$$

Lagrange equations:

x_2 : $$\left(m_2+\frac{m_1}{4}\right)\ddot{x}_2+\frac{m_1}{4}\ddot{x}_3=g\left(m_2-\frac{m_1}{2}\right)$$

x_3 : $$\left(m_3+\frac{m_1}{4}\right)\ddot{x}_3+\frac{m_1}{4}\ddot{x}_2=g\left(m_3-\frac{m_1}{2}\right)$$

These can be solved by elimination, e.g.

$$\ddot{x}_2=-\frac{m_1}{m_1+4m_2}x_3+g\frac{m_2-\frac{m_1}{2}}{m_2+\frac{m_1}{4}}=-\frac{m_1}{m_1+4m_2}\ddot{x}_3+\frac{4m_2-2m_1}{m_1+4m_2}g$$

$$\left(m_3+\frac{m_1}{4}\right)\ddot{x}_3+\frac{m_1}{4}\left(-\frac{m_1}{m_1+4m_2}\ddot{x}_3+g\frac{4m_2-2m_1}{m_1+4m_2}\right)=g\left(m_3-\frac{m_1}{2}\right)$$

$$\underbrace{\left[m_3+\frac{m_1}{4}\left(1-\frac{m_1}{m_1+4m_2}\right)\right]}_{m_3+\frac{m_1m_2}{m_1+4m_2}}\ddot{x}_3=g\left(m_3-\frac{m_1}{2}-\frac{m_1}{4}\,\frac{4m_2-2m_1}{m_1+4m_2}\right)$$

$$\left[m_3(m_1+4m_2)+m_1m_2\right]\ddot{x}_3=g\left[\left(m_3-\frac{m_1}{2}\right)(m_1+4m_2)-m_1\left(m_2-\frac{m_1}{2}\right)\right]$$

$$\ddot{x}_3=g\,\frac{m_1m_3+4m_2m_3-3m_1m_2}{m_1m_3+4m_2m_3+m_1m_2}$$

Due to the symmetry of the Lagrange equations, $\ddot{x}_2$ is obtained from the last expression upon $m_2 \rightleftarrows m_3$:

$$\ddot{x}_2=g\,\frac{m_1m_2+4m_2m_3-3m_1m_3}{m_1m_2+4m_2m_3+m_1m_3}$$

Finally, $\ddot{x}_1=-\frac{1}{2}\left(\ddot{x}_2+\ddot{x}_3\right)$ and

$$\ddot{x}_1=-\frac{g}{2}\left(\frac{m_1m_2+4m_2m_3-3m_1m_3}{m_1m_2+4m_2m_3+m_1m_3}+\frac{m_1m_3+4m_2m_3-3m_1m_2}{m_1m_3+4m_2m_3+m_1m_2}\right)=$$

$$=g\,\frac{m_1(m_2+m_3)-4m_2m_3}{m_1(m_2+m_3)+4m_2m_3}$$

② If the horizontal displacement of the block is X then the position of the bob is given by

$$x = X + l\sin\theta \qquad y = -l\cos\theta$$

where θ is the angle between the rod and the vertical axis. Then the Lagrangian of the system is

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy =$$

$$= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X}^2 + 2l\dot{X}\dot{\theta}\cos\theta + l^2\dot{\theta}^2) + mgl\cos\theta$$

Coordinate X is cyclic.

$$\frac{\partial L}{\partial X} = 0 \qquad \frac{\partial L}{\partial \dot{X}} = M\dot{X} + m\dot{X} + ml\dot{\theta}\cos\theta$$

$$\frac{\partial L}{\partial \theta} = -ml\dot{X}\dot{\theta}\sin\theta - mgl\sin\theta \qquad \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + ml\dot{X}\cos\theta$$

The Lagrange equations are

$$(m+M)\ddot{X} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = 0$$

$$ml^2\ddot{\theta} + ml\ddot{X}\cos\theta \cancel{- ml\dot{X}\dot{\theta}\sin\theta} = \cancel{-ml\dot{X}\dot{\theta}\sin\theta} - mgl\sin\theta$$

When θ is small these equations simplify to

$$\begin{cases} (m+M)\ddot{X} + ml\ddot{\theta} = 0 \\ ml^2\ddot{\theta} + ml\ddot{X} + mgl\theta = 0 \end{cases} \quad \text{or} \quad \begin{aligned} &\ddot{X} = -\frac{m}{m+M}l\ddot{\theta} \\ &ml^2\ddot{\theta} + ml^2\left(-\frac{m}{m+M}\right)\ddot{\theta} + mgl\theta = 0 \end{aligned}$$

$$\ddot{\theta} + \frac{m+M}{M}\frac{g}{l}\theta = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{g(M+m)}{lM}}$$

$$\theta = A\sin\omega t + B\cos\omega t$$

$$X = C\sin\omega t + D\cos\omega t$$

(3) Since the light starts propagating along the y axis and n is a function of z coordinate only, the path will lie in yz plane (so ~~that~~ that we can ignore x coordinate). The total travel time is given by the following integral over the path of light

$$T = \int \frac{ds}{v}$$

Now $\quad ds = \sqrt{1+z_y'^2}\,dy \qquad \frac{1}{v} = \frac{n}{c} = (1+\alpha z)\frac{n_0}{c}$

Hence the path $z(y)$ has to minimize the functional

$$T = \frac{n_0}{c}\int \sqrt{1+z_y'^2}\,(1+\alpha z)\,dy \qquad \text{with } F(z,z',y) = \sqrt{1+z_y'^2}\,(1+\alpha z)$$

$$\delta T = 0 \implies \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial z} \quad \leftarrow \text{Euler-Lagrange equation}$$

$$\frac{\partial F}{\partial z} = \alpha\sqrt{1+z_y'^2} \qquad \frac{\partial F}{\partial z'} = (1+\alpha z)\frac{z'}{\sqrt{1+z'^2}}$$

$$\frac{d}{dy}\left((1+\alpha z)\frac{z'}{\sqrt{1+z'^2}}\right) = \alpha\sqrt{1+z'^2}$$

Solving this differential equation would give $z(y)$. However, it is not immediately obvious how to solve it. It is going to be an easier task if we switch the roles of dependent and independent variables in our problem and look for $y(z)$ rather than $z(y)$.

$$ds = \sqrt{1+y_z'^2}\,dz$$

$$T = \frac{n_0}{c}\int \sqrt{1+y_z'^2}\,(1+\alpha z)\,dz \qquad \text{with } G(y,y',z) = \sqrt{1+y_z'^2}\,(1+\alpha z)$$

Then the Euler-Lagrange equation is

$$\frac{\partial G}{\partial y} = 0 \qquad \frac{\partial G}{\partial y'} = (1+\alpha z)\frac{y'}{\sqrt{1+y_z'^2}}$$

$$\frac{d}{dz}\left((1+\alpha z)\frac{y'}{\sqrt{1+y_z'^2}}\right) = 0 \quad \text{or} \quad \frac{(1+\alpha z)y'}{\sqrt{1+y_z'^2}} = \text{const} = b$$

or $$(1+\alpha z)^2 y'^2 = b^2(1+y_z'^2)$$

Then we can solve for y_z'

$$\frac{dy}{dz} = \frac{b^2}{\sqrt{(1+\alpha z)^2 - b^2}}$$

$$\int dy = b^2 \int \frac{dz}{\sqrt{(1+\alpha z)^2 - b^2}} = \frac{b^2}{\alpha}\int \frac{dz}{\sqrt{(z+\frac{1}{\alpha})^2 - \frac{b^2}{\alpha^2}}}$$

$$z + \frac{1}{\alpha} = q \qquad dz = dq \qquad \gamma = \frac{b}{\alpha} \qquad \frac{q}{\gamma} = u$$

$$y + k = \frac{b^2}{\alpha}\int \frac{dq}{\sqrt{q^2-\gamma^2}} = \frac{b^2}{\alpha}\int \frac{d\frac{q}{\gamma}}{\sqrt{(\frac{q}{\gamma})^2 - 1}} = \frac{b^2}{\alpha}\int \frac{du}{\sqrt{u^2-1}}$$

$$= \frac{b^2}{\alpha}\ln\left[u + \sqrt{u^2-1}\right] = \frac{b^2}{\alpha}\operatorname{arccosh}(u) = \frac{b^2}{\alpha}\operatorname{arccosh}\left(\frac{z+\frac{1}{\alpha}}{\frac{b}{\alpha}}\right) =$$

$$= \frac{b^2}{\alpha}\operatorname{arccosh}\left(\frac{\alpha z + 1}{b}\right)$$ where k is a constant

Solving for z gives

$$z = \frac{1}{\alpha}\left[b\cosh\left(\frac{\alpha}{b^2}(y+k)\right) - 1\right]$$

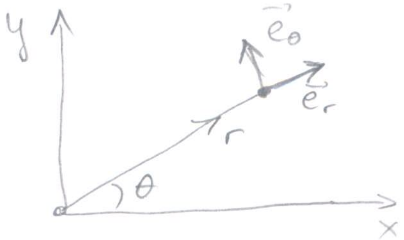
Constants b and k can be determined from the condition $z(y=0)=0$ and $z_y'(y=0)=0$

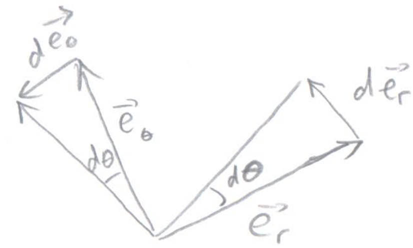
$$b\cosh\left(\frac{\alpha k}{b^2}\right) - 1 = 0 \quad \text{and} \quad \frac{1}{b}\sinh\left(\frac{\alpha k}{b^2}\right) = 0 \implies \begin{matrix} k=0 \\ b=1 \end{matrix}$$

(4) To check if the force is central we can see if its transverce component is zero.

Let us first consider acceleration in polar coordinate system:

$$\vec{r} = r\vec{e}_r$$





$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\vec{e}_r + r\frac{d\vec{e}_r}{dt} \qquad d\vec{e}_r = \vec{e}_\theta d\theta \qquad \frac{d\vec{e}_r}{dt} = \vec{e}_\theta\frac{d\theta}{dt} = \vec{e}_\theta\dot{\theta}$$

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\vec{e}_r + \dot{r}\frac{d\vec{e}_r}{dt} + \dot{r}\dot{\theta}\vec{e}_\theta + r\ddot{\theta}\vec{e}_\theta + r\dot{\theta}\frac{d\vec{e}_\theta}{dt}$$

$$d\vec{e}_\theta = -\vec{e}_r d\theta \qquad \frac{d\vec{e}_\theta}{dt} = -\vec{e}_r\frac{d\theta}{dt}$$

Hence

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + \underbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta}_{\text{transverse component}}$$

$$F_\theta = m a_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

For $r = \alpha\theta$ where $\theta = \beta t$ (β is some constant)

we have

$$F_\theta = m(\alpha\theta\underbrace{(\ddot{\beta t})}_{0} + 2\alpha(\dot{\beta t})(\dot{\beta t})) = 2m\alpha\beta^2 \neq 0$$

Therefore the force is not central

Now let us try the dependence $r = \alpha\theta$ $\theta = \beta t^n$

$$F_\theta = m\left[\alpha\beta t^n n(n-1)\beta t^{n-2} + 2\alpha\beta n t^{n-1}\beta n t^{n-1}\right] = 0 \Rightarrow n(n-1) = 2n^2$$

$n = 1/$[illegible] and it must be that $\theta = \beta t^{\frac{1}{3}}$