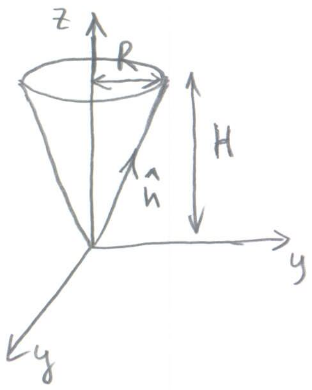
(1) Let us first compute the components of the tensor of inertia in the reference frame where the z-axis coincides with the symmetry axis of the cone and the origin is at the apex. Due to the symmetry the only nonzero components are



$$I_{xx} = I_{yy} \quad \text{and} \quad I_{zz}.$$

The density of the cone is $\rho = \frac{M}{V} = \frac{M}{\frac{1}{3}\pi R^2 H} = \frac{3M}{\pi R^2 H}$

$$I_{xx} = \int_V \rho(y^2+z^2)\,dV \qquad I_{zz} = \int \rho(x^2+y^2)\,dV$$

$$\rho\int_V y^2\,dV = \rho\int_0^{2\pi} d\theta \int_0^R r\,dr \int_{\frac{r}{R}H}^{H} dz\, y^2 = \rho \underbrace{\int_0^{2\pi} \sin^2\theta\, d\theta}_{\pi} \int_0^R r^3 dr \int_{\frac{r}{R}H}^{H} dz =$$

$$= \pi\rho\int_0^R r^3 dr\left(H - \frac{r}{R}H\right) = \pi\rho H\left(\int_0^R r^3 dr - \frac{1}{R}\int_0^R r^4 dr\right) = \frac{\pi\rho H R^4}{20} = \frac{3MR^2}{20}$$

$$\rho\int_V z^2\,dV = \rho\int_0^{2\pi} d\theta \int_0^R r\,dr \int_{\frac{r}{R}H}^{H} dz\, z^2 = \frac{2\pi\rho}{3}\int_0^R r\,dr\left(H^3 - \frac{r^3}{R^3}H^3\right) = \frac{2\pi\rho H^3}{3}\left(\int_0^R r\,dr - \frac{1}{R^3}\int_0^R r^4 dr\right) =$$

$$= \frac{2\pi\rho H^3}{3}\left(\frac{R^2}{2} - \frac{R^2}{5}\right) = \frac{\pi\rho H^3 R^2}{5} = \frac{3}{5}MH^2$$

Then

$$I_{xx} = I_{yy} = \frac{3MR^2}{20} + \frac{3}{5}MH^2 \qquad I_{zz} = \frac{3MR^2}{10}$$

The components of unit vector $\hat{n}$ that lies on the surface of the cone in yz-plane are

$$\hat{n} = \frac{1}{\sqrt{R^2+H^2}}\begin{pmatrix} 0 \\ R \\ H \end{pmatrix}$$

The moment of inertia about the axis defined by this unit vector for $\hat{n}$ is

$$I_{\vec{n}} = \hat{n}^T \cdot I \cdot \hat{n} = \frac{1}{R^2+H^2}(0\ R\ H)\begin{pmatrix} \frac{3MR^2}{20}+\frac{3}{5}MH^2 & 0 & 0 \\ 0 & \frac{3MR^2}{20}+\frac{3}{5}MH^2 & 0 \\ 0 & 0 & \frac{3MR^2}{10} \end{pmatrix}\begin{pmatrix} 0 \\ R \\ H \end{pmatrix} =$$

$$= \frac{M}{R^2+H^2}\left(\frac{3R^4}{20} + \frac{3}{5}H^2R^2 + \frac{3R^2H^2}{10}\right) = \frac{MR^2}{R^2+H^2}\left(\frac{3R^2}{20} + \frac{9}{10}H^2\right)$$

(2) The Euler equations that describe the motion of a rigid body are

$$\begin{cases} I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = N_1 \\ I_2 \dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = N_2 \\ I_3 \dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = N_3 \end{cases}$$

Since $I_2 = I_1$ the third equation is simplified (and gets decoupled from the other two):

$$I_3 \dot{\omega}_3 = N_3$$

Now $N_3 = -b\omega_3$ (because $\vec{N}_f = -b\vec{\omega}$), so we get a first order equation

$$\dot{\omega}_3 = -\frac{b}{I_3}\omega_3 \quad \Longrightarrow \quad \frac{d\omega_3}{\omega_3} = -\frac{b}{I_3}dt$$

$$\omega_3(t) = C e^{-\frac{b}{I_3}t} \quad \text{where} \quad C = \omega_3(0)$$

(3) Kinetic energy: $T = \frac{m\dot{x}^2}{2} + \frac{I\omega^2}{2}$ where

$\omega = \frac{\dot{x}}{R}$ and $I = \frac{1}{2}MR^2$ (here R is the radius of the cylider)

so $\frac{I\omega^2}{2} = \frac{M\dot{x}^2}{4}$

Potential energy: $V = -mgx + \frac{kx^2}{2}$

Lagrangian: $L = \frac{1}{2}(m + \frac{M}{2})\dot{x}^2 + mgx - \frac{kx^2}{2}$

Equation of motion:

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$ or $(m + \frac{M}{2})\ddot{x} + kx - mg = 0$

If we substitute $y = x + \frac{mg}{k}$ we get

$$(m + \frac{M}{2})\ddot{y} + ky = 0$$

This is a standard harmonic oscillator equation, so the frequency of oscillations is

$$\omega = \sqrt{\frac{k}{m + \frac{M}{2}}}$$

(4) Kinetic energy: $T = m\ell^2\dot{\theta}_1^2 + \frac{m}{2}\ell^2\dot{\theta}_2^2$

Potential energy: $V = -2mg\ell\cos\theta_1 - mg\ell\cos\theta_2 + \frac{k\ell^2}{2}(\sin\theta_2 - \sin\theta_1)^2$

For small θ_1 and θ_2 V can be written as

$$V = mg\ell\,\theta_1^2 + \frac{1}{2}mg\ell\,\theta_1^2 + \frac{k\ell^2}{2}(\theta_2 - \theta_1)^2$$

Hence the Lagrangian for small oscillations becomes

$$L = m\ell^2\dot{\theta}_1^2 + \frac{m}{2}\ell^2\dot{\theta}_2^2 - mg\ell\,\theta_1^2 - \frac{1}{2}mg\ell\,\theta_2^2 - \frac{k\ell^2}{2}(\theta_2 - \theta_1)^2$$

The Lagrange equations for θ_1 and θ_2 are (after we divide everything by $m\ell^2$

$$2\ddot{\theta}_1 = -2\frac{g}{\ell}\theta_1 + \frac{k}{m}(\theta_2 - \theta_1)$$

$$\ddot{\theta}_1 = -\frac{g}{\ell}\theta_2 - \frac{k}{m}(\theta_2 - \theta_1)$$

or in the matrix form

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\begin{pmatrix} 2\gamma + \varkappa & -\varkappa \\ -\varkappa & \gamma + \varkappa \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

where $\gamma \equiv \frac{g}{\ell}$ and $\varkappa \equiv \frac{k}{m}$

If we look for the solution as $\vec{\theta} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$ then we obtain the eigenvalue problem for ω^2:

$$\begin{pmatrix} 2\gamma + \varkappa & -\varkappa \\ -\varkappa & \gamma + \varkappa \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \omega^2 \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

The eigenvalues are the roots of the secular equation:

$$\begin{vmatrix} 2\gamma + \varkappa - 2\omega^2 & -\varkappa \\ -\varkappa & \gamma + \varkappa - \omega^2 \end{vmatrix} = 0 \qquad (2\gamma + \varkappa - 2\omega^2)(\gamma + \varkappa - \omega^2) - \varkappa^2 = 0$$

or

$$2\omega^4 - (4\gamma + 3\varkappa)\omega^2 + 3\gamma\varkappa + 2\gamma^2 = 0$$

The latter is a quadratic equation with respect to ω^2

$$D = \sqrt{(4\gamma + 3\varkappa)^2 - 8(3\gamma\varkappa + 2\gamma^2)} = \sqrt{\cancel{16\gamma^2} + 9\varkappa^2 + \cancel{24\gamma\varkappa} - \cancel{24\gamma\varkappa} - \cancel{16\gamma^2}} = 3\varkappa$$

so

$$\omega_{1,2}^2 = \frac{4\gamma + 3\varkappa \pm 3\varkappa}{4} \Rightarrow \omega_1 = \sqrt{\gamma} = \sqrt{\frac{g}{\ell}} \qquad \omega_2 = \sqrt{\gamma + \frac{3}{2}\varkappa} = \sqrt{\frac{g}{\ell} + \frac{3}{2}\frac{k}{m}}$$

Now let us find the eigenvector corresponding to ω_1^2

$$\begin{pmatrix} 2\gamma+\varkappa-2\gamma & -\varkappa \\ -\varkappa & \gamma+\varkappa-\gamma \end{pmatrix}\begin{pmatrix} a_1^{(1)} \\ a_2^{(1)} \end{pmatrix} = 0$$

$\varkappa a_1^{(1)} - \varkappa a_2^{(1)} = 0$

$a_1^{(1)} = a_2^{(1)}$

Both pendula move in phase

So the first normal mode is

$$\theta(t) = \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = A \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} e^{i\omega_1 t}$$

The eigenvector corresponding to ω_2^2 is

$$\begin{pmatrix} 2\gamma+\varkappa-2(\gamma+\frac{3}{2}\varkappa) & -\varkappa \\ -\varkappa & \gamma+\varkappa-(\gamma+\frac{3}{2}\varkappa) \end{pmatrix}\begin{pmatrix} a_1^{(2)} \\ a_2^{(2)} \end{pmatrix} = 0$$

$$-\varkappa a_1^{(2)} - \frac{1}{2}\varkappa a_2^{(2)} = 0 \qquad a_2^{(2)} = -2a_1^{(2)}$$

The second normal mode is

$$\theta(t) = \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = A \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} e^{i\omega_2 t}$$

In the second normal mode pendula move in opposite phase while the amplitude of the first one is twice smaller