Name: Mr. Key

PHYS 505: Classical Mechanics (graduate), Quiz #1

Instruction: use additional sheets if you find it necessary

1. Find the Taylor series expansion of $\sin x$ at point $x = \pi/2$ up to the fourth order.

2. Find the value of the following integral when $a \to 0$:

$$\int_0^a \frac{x^2}{1 - \exp(-\beta x^2)} dx$$

(1) There are two ways to find the Taylor series. The first one is to follow the definition:

$$f(x - x_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

In our case $f(x) = \sin x$ $x_0 = \frac{\pi}{2}$

Then

$\sin^{(0)}\frac{\pi}{2} = 1$ $\quad \sin^{(1)}\frac{\pi}{2} = \cos\frac{\pi}{2} = 0$ $\quad \sin^{(2)}\frac{\pi}{2} = -\sin\frac{\pi}{2} = -1$

$\sin^{(3)}\frac{\pi}{2} = -\cos\frac{\pi}{2} = 0$ $\quad \sin^{(4)}\frac{\pi}{2} = \sin\frac{\pi}{2} = 1$

and

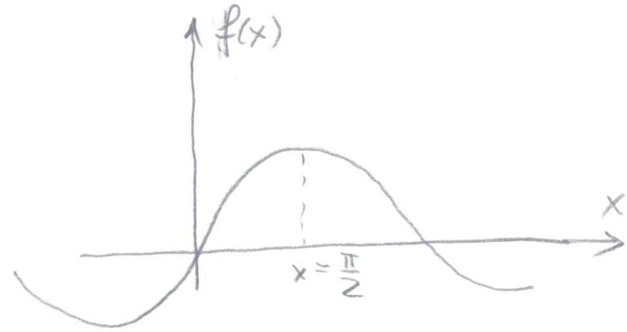
$$\sin x \overset{x \to \frac{\pi}{2}}{=} 1 - \frac{(x - \frac{\pi}{2})^2}{2!} + \frac{(x - \frac{\pi}{2})^4}{4!} - \dots$$

The second and easier way is to recall that

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x$$

When $x \to 0$ we know the expansion for $\cos x$:

$$\cos x \overset{x \to 0}{=} 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$



Now we just replace

$x \Rightarrow x - \frac{\pi}{2}$ and get $\quad \sin x \overset{x \to \frac{\pi}{2}}{=} 1 - \frac{(x - \frac{\pi}{2})^2}{2!} + \frac{(x - \frac{\pi}{2})^4}{4!} - \dots$

(2) When $a \to 0$ x in the integrand is always small, so we can approximate it with its Maclaurin series and integrate each term separately:

$$\frac{x^2}{1-e^{-\beta x^2}} \overset{x\to 0}{=} \frac{x^2}{1-\left(1-\beta x^2+\frac{\beta^2 x^4}{2!}-\frac{\beta^3 x^6}{3!}+\dots\right)} = \frac{x^2}{\beta x^2-\frac{\beta^2 x^4}{2}+\dots}$$

$$\int_0^a \frac{x^2}{1-e^{-\beta x^2}}\,dx \overset{a\to 0}{=} \int_0^a \left(\frac{1}{\beta}+O(x^2)\right)dx = \frac{a}{\beta}+O(a^3)$$