① Here we need to find the minimum and maximum of function $f(x,y,z) = x^2+y^2+z^2$ subject to constraints $g(x,y,z) = x^2+y^2-z=0$ and $h = x+y+2z-2=0$. We form the Lagrange function $L = f - \lambda g - \mu h$ that has two undetermined multipliers and require that $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = \frac{\partial L}{\partial \lambda} = \frac{\partial L}{\partial \mu} = 0$. This yields

$$\begin{cases} 2x - 2\lambda x - \mu = 0 \\ 2y - 2\lambda y - \mu = 0 \\ 2z + \lambda - 2\mu = 0 \\ x^2+y^2-z=0 \\ x+y+2z-2=0 \end{cases}$$

From eq. #3 it follows that $\lambda = 2\mu - 2z$ and if we eliminate λ we get

$$\begin{cases} 2x - 4\mu x + 4zx - \mu = 0 \\ 2y - 4\mu y + 4zy - \mu = 0 \\ x^2+y^2-z=0 \\ x+y+2z-2=0 \end{cases}$$

Then we eliminate μ: $\mu = \frac{2x+4zx}{1+4x}$

$$2y+4zy = \frac{2x+4zx}{1+4x}(1+4y) \Rightarrow (1+4x)(1+2z)y = (1+4y)(1+2z)x$$

or $(1+4x)y = (1+4y)x \Rightarrow x = y$

Then the last two equations in the system simplify to $z = 2x^2$ and $z = 1-x$ which gives a quadratic equation $2x^2+x-1=0 \Rightarrow x_{1,2} = -1, \frac{1}{2} \Rightarrow y_{1,2} = -1, \frac{1}{2}$

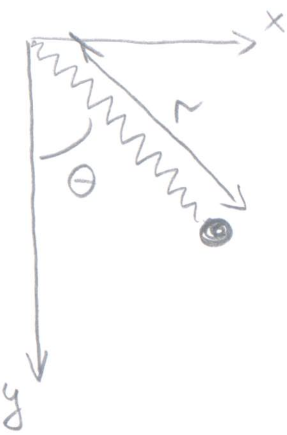
$z_{1,2} = 2, \frac{1}{2}$. So the extema are located at

$(-1,-1,2)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

$f(-1,-1,2) = 6$ — max $\qquad f(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = \frac{3}{4}$ — min

(2) It is convenient to use polar coordinates here

$$\begin{cases} x = r\sin\theta \\ y = r\cos\theta \end{cases} \qquad \begin{cases} \dot{x} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ \dot{y} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \end{cases}$$



The kinetic energy is then

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)$$

The potential energy is

$$V = -mgy + \frac{k}{2}(r-r_0)^2 = -mgr\cos\theta + \frac{k}{2}(r-r_0)^2$$

where r_0 is the length of the unstretched string

The Lagrangian is then

$$L = T - V = \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2) + mgr\cos\theta - \frac{k}{2}(r-r_0)^2$$

The Lagrange equations follow immediately:

$$\begin{cases} \frac{d}{dt}(mr^2\dot{\theta}) + mgr\sin\theta = 0 \\ \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 - mg\cos\theta + k(r-r_0) \end{cases}$$

or

$$\begin{cases} r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta = 0 \\ \ddot{r} - r\dot{\theta}^2 - g\cos\theta + \frac{k}{m}(r-r_0) = 0 \end{cases}$$

If $\theta \to 0$ then $\sin\theta \to \theta$ and $\cos\theta \to 1$. Then the above equations become

$$\begin{cases} r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\theta = 0 \\ \ddot{r} - r\dot{\theta}^2 - g + \frac{k}{m}(r-r_0) = 0 \end{cases}$$

Further, since we are considering the motion near the equilibrium we can assume that all generalized velocities are small, i.e. we can neglect the terms containing first derivatives, which yields

$$\begin{cases} r\ddot{\theta} + g\theta = 0 \\ \ddot{r} - g + \frac{k}{m}(r - r_0) = 0 \end{cases}$$

For r ~~at~~ the equilibrium we can write

$$r(t) = r_{eq} + \xi(t)$$

where r_{eq} is the equilibrium length of the spring when the ball is suspended vertically. Obviously

$$r_{eq} = r_0 + \frac{mg}{k}$$

Then the equations of motion become

$$\begin{cases} \ddot{\theta} + \frac{g}{r_{eq}}\theta = 0 \\ \ddot{\xi} + \frac{k}{m}\xi = 0 \end{cases}$$

The solution is

$$\theta(t) = A\sin\omega t + B\cos\omega t \qquad \omega = \sqrt{\frac{g}{r_{eq}}}$$

$$\xi(t) = C\sin\Omega t + D\cos\Omega t \qquad \Omega = \sqrt{\frac{k}{m}}$$

where $A, B, C,$ and D are determined from the initial conditions.

Variables are separated and the small oscillations along the r (ξ) and θ coordinates occur independently