Here we need to diagonalize I, i.e. find its eigenvalues and eigenvectors. The eigenvalues (the diagonal elements of I when it is reduced to a diagonal form) are the principle moments of inertia. The eigenvectors will give the axes of rotation such that tho moment of inertia is I_1, I_2, or I_3 when the body is rotated abent those axes

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^3 - (2-\lambda) - (2-\lambda) = 0$$

$$(2-\lambda)\left[(2-\lambda)^2 - 2\right] = 0$$

$\lambda_1 = 2$

$$2 - 4\lambda + \lambda^2 = 0$$

$$\lambda_{2,3} = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$$

So $I_1 = 2$ $\quad I_2 = 2+\sqrt{2}$ $\quad I_3 = 2-\sqrt{2}$

The eigenvector corresponding to $\lambda_1 = 2$ is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{array}{l} 2x - y = 2x \\ -x + 2y - z = y \\ -y + 2z = 2z \end{array} \qquad \begin{array}{l} y = 0 \\ z = -x \end{array}$$

so $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ the normalized egenvector is $\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

The eigenvector corresponding to $\lambda_2 = 2+\sqrt{2}$ is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (2+\sqrt{2})\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$2x - y = (2+\sqrt{2})x \qquad y = -\sqrt{2}x$$

$$-x + 2y - z = (2+\sqrt{2})y$$

$$-y + 2z = (2+\sqrt{2})z \qquad z = -\frac{y}{\sqrt{2}} = x$$

So $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$ and the normalized eigenvector is $\vec{u}_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$

Finally, the eigenvector corresponding to $\lambda_3 = 2-\sqrt{2}$ is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (2-\sqrt{2})\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$2x - y = (2-\sqrt{2})x \qquad y = \sqrt{2}x$$

$$-x + 2y - z = (2-\sqrt{2})y$$

$$-y + 2z = (2-\sqrt{2})z \qquad z = \frac{y}{\sqrt{2}} = x$$

So

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$ and the normalized eigenvector is $\vec{u}_3 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$

It is easy to check that all eigenvectors $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ are mutually orthogonal, which is expected given the fact that I is a symmetric matrix.